Areas of Limaçons

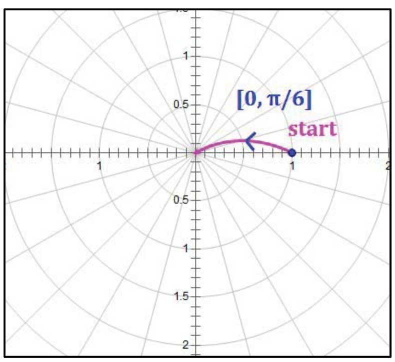
Limaçons that have both inner and outer loops present a challenge when calculating area. The general form of a limaçon is:

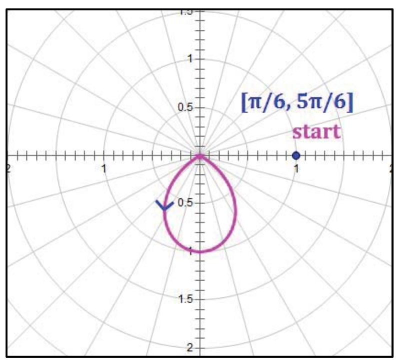
$$r = a + b\cos\theta \qquad \text{or} \qquad r = a + b\sin\theta$$

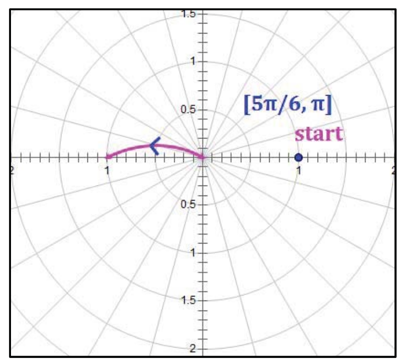
When $|a| < |b|$, the limaçon has an inner loop that covers part of its outer loop, so we must be careful calculating areas in this kind of limaçon.

Example 8: Find the area between the loops (i.e., inside the outer loop but outside the inner loop) of the limaçon: $r = 1 - 2\sin\theta$.

First, we need to find where $r = 1 - 2\sin\theta = 0$ so we can identify the starting and ending θ-values for the inner loop. After finding these values to be $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, we can look at the curve over various intervals on $[0, 2\pi]$ and calculate the areas associated with those intervals.





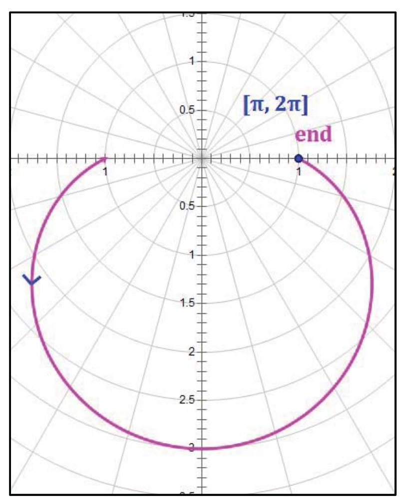


$$\left[0, \frac{\pi}{6}\right]: \quad \frac{1}{2}\int_0^{\pi/6} (1 - 2\sin\theta)^2\, d\theta = \frac{\pi + 3\sqrt{3} - 8}{4} \sim 0.0844$$

$$\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]: \quad \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 2\sin\theta)^2\, d\theta = \frac{2\pi - 3\sqrt{3}}{2} \sim 0.5435$$

$$\left[\frac{5\pi}{6}, \pi\right]: \quad \frac{1}{2}\int_{5\pi/6}^{\pi} (1 - 2\sin\theta)^2\, d\theta = \frac{\pi + 3\sqrt{3} - 8}{4} \sim 0.0844$$

$$[\pi, 2\pi]: \quad \frac{1}{2}\int_{\pi}^{2\pi} (1 - 2\sin\theta)^2\, d\theta = \frac{3\pi + 8}{2} \sim 8.7124$$



The total area of the limaçon, including both the outer and inner loops, is the sum of these:

$$[0, 2\pi]: \quad \frac{1}{2}\int_0^{2\pi} (1 - 2\sin\theta)^2\, d\theta = 3\pi \sim 9.4248$$

A sketch of the complete limaçon $r = 1 - 2\sin\theta$ is shown in Figure 1 below. Since taking the area from 0 to 2π includes the area completely inside the outer loop plus the area inside the inner loop, the total area can be thought of as shown in Figure 2.

This illustrates that the area within the inner loop is included in $A = \frac{1}{2}\int_0^{2\pi}(1-2\sin\theta)^2\,d\theta$ twice, and therefore, must be subtracted twice when looking for the area between the loops. Subtracting it once leaves all of the area inside the outer loop (Figure 3). A second subtraction is required to obtain the area between the loops.

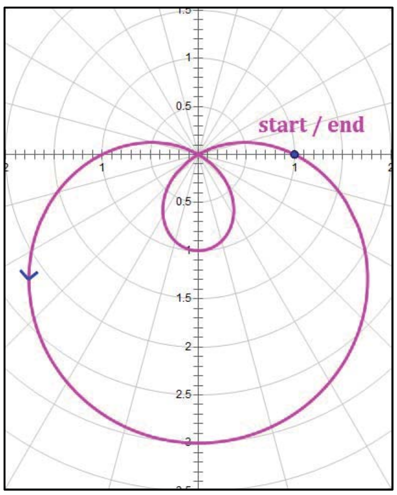


Figure 1
$r = 1 - 2\sin\theta$
Graphed on $[0, 2\pi]$

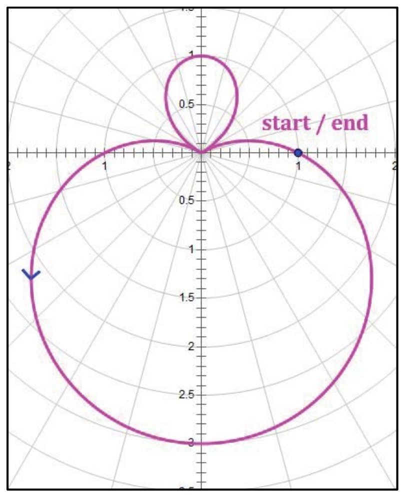


Figure 2
$r = |1 - 2\sin\theta|$
Graphed on $[0, 2\pi]$

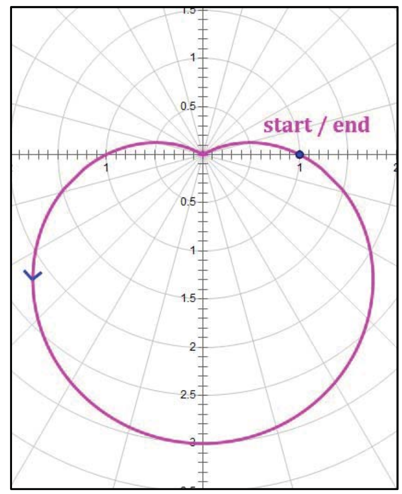


Figure 3
$r = 1 - 2\sin\theta$
Graphed on $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$

Given all of the above, let's calculate the key areas of the limaçon $r = 1 - 2\sin\theta$:

The total area of the limaçon, including both the outer loop and the inner loop, is:

$$\text{Interval } [0,2\pi]:\quad \frac{1}{2}\int_0^{2\pi}(1-2\sin\theta)^2\,d\theta \;=\; 3\pi \;\sim\; 9.4248$$

The area inside the inner loop is calculated as:

$$\text{Interval } \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]:\quad \frac{1}{2}\int_{\pi/6}^{5\pi/6}(1-2\sin\theta)^2\,d\theta \;=\; \frac{2\pi - 3\sqrt{3}}{2} \;\sim\; 0.5435$$

The area between the loops (i.e., the solution to this example) is calculated as:

$$\frac{1}{2}\int_0^{2\pi}(1-2\sin\theta)^2\,d\theta \;-\; 2\cdot\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(1-2\sin\theta)^2\,d\theta \;=\; \pi + 3\sqrt{3} \;\sim\; 8.3377$$