Part I: Multiple Choice (no calculator)

1) If \( y = x \sin x \), then \( \frac{dy}{dx} = \)

A) \( \sin x + \cos x \)  
B) \( \sin x + x \cos x \)  
C) \( \sin x - x \cos x \)  
D) \( x(\sin x + \cos x) \)  
E) \( x(\sin x - \cos x) \)

\[ y = x \sin x \]
\[
\frac{dy}{dx} = x \cdot \frac{d}{dx} \sin x + \sin x \cdot \frac{d}{dx} x
\]
\[ = x \cos x + \sin x \]

Answer B

2) If \( f(x) = 7x - 3 + \ln x \), then \( f'(1) = \)

A) 4  
B) 5  
C) 6  
D) 7  
E) 8

\[ f(x) = 7x - 3 + \ln x \]
\[ f'(x) = 7 - 0 + \frac{1}{x} \]
\[ f'(1) = 7 + \frac{1}{1} \]
\[ = 8 \]

Answer E

3) The graph of function \( f \) is shown. Which of the following statements is false?

(A) \( \lim_{x \to 2} f(x) \) exists.  
(B) \( \lim_{x \to -3} f(x) \) exists.  
(C) \( \lim_{x \to 4} f(x) \) exists.  
(D) \( \lim_{x \to 5} f(x) \) exists.  
(E) The function \( f \) is continuous at \( x = 3 \).

Comments:

(A) TRUE. The limit exists because the function approaches the same value from the left and right. Note that the value of the function \( f(2) = 1 \) is different than the value of the limit \( \lim_{x \to 2} f(x) = 2 \). So, as an aside, we note that the function is not continuous at \( x = 2 \).

(B) TRUE. The limit exists because the function approaches the same value from the left and right at \( x = 3 \).

(C) FALSE. The limit does not exist because the function approaches different values from the left and right at \( x = 4 \).

(D) TRUE. The limit exists because the function approaches the same value from the left and right at \( x = 5 \).

(E) TRUE. The function is continuous because a) the function approaches the same value from the left and right at \( x = 3 \) and b) that limit is the value of the function at \( x = 3 \).
4) If \( y = (x^3 - \cos x)^5 \), then \( y' = \) 
\[ y' = 5(x^3 - \cos x)^4 \cdot \frac{d}{dx}(x^3 - \cos x) \]

\( = 5(x^3 - \cos x)^4 \cdot (3x^2 + \sin x) \)

\( \text{Answer E} \)

5) \( f(x) = \begin{cases} \frac{(2x+1)(x-2)}{x-2} & \text{for } x \neq 2 \\ k & \text{for } x = 2 \end{cases} \)

Let \( f \) be the function defined above. For what value of \( k \) is \( f \) continuous at \( x = 2 \)?

A) 0
B) 1
C) 2
D) 3
E) 5

To get the value of \( k \) at \( x = 2 \) for which the function is continuous, calculate the limit as \( x \to 2 \).

\[ \lim_{x \to 2} f(x) = \lim_{x \to 2} \left( \frac{(2x+1)(x-2)}{x-2} \right) \]

\[ = \lim_{x \to 2} (2x + 1) \]

\[ = (2 \cdot 2 + 1) = 5 \]

\( \text{Answer E} \)
6) If \( f(x) = \sqrt{x^2 - 4} \) and \( g(x) = 3x - 2 \), then the derivative of \( f(g(x)) \) at \( x = 3 \) is

(A) \( \frac{7}{5} \)  \hspace{1cm} (B) \( \frac{14}{\sqrt{5}} \)  \hspace{1cm} (C) \( \frac{18}{\sqrt{5}} \)  \hspace{1cm} (D) \( \frac{15}{\sqrt{21}} \)  \hspace{1cm} (E) \( \frac{30}{\sqrt{21}} \)

Use the chain rule:

\[ f'(g(x))' = f'(g(x)) \cdot g'(x) \]

First, find \( f'(g(x)) \):

\[ f'(x) = \frac{d}{dx} (x^2 - 4)^{1/2} = \frac{1}{2} \cdot (x^2 - 4)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 - 4}} \]

\[ f'(g(x)) = \frac{3x - 2}{\sqrt{(3x - 2)^2 - 4}} \]

Then, find \( g'(x) \):

\[ g'(x) = \frac{d}{dx} (3x - 2) = 3 \]

Finally, substitute \( x = 3 \) into the chain rule expression for: \( f'(g(x))' = f'(g(x)) \cdot g'(x) \)

\[ f'(g(x)) \bigg|_{x=3} = \frac{(3x-2)}{\sqrt{(3x-2)^2 - 4}} \cdot 3 = \frac{(3\cdot3-2)}{\sqrt{(3\cdot3-2)^2 - 4}} \cdot 3 = \frac{7}{\sqrt{45}} \cdot 3 = \frac{7}{3\sqrt{5}} \cdot 3 = \frac{7}{\sqrt{5}} \]

Answer A

7) \[ \lim_{h \to 0} \frac{\ln(4 + h) - \ln(4)}{h} \]

(A) 0  \hspace{1cm} (B) \( \frac{1}{4} \)  \hspace{1cm} (C) 1  \hspace{1cm} (D) \( e \)  \hspace{1cm} (E) nonexistent

Notice that the expression is the definition of a derivative:

\[ \lim_{h \to 0} \frac{\ln(4 + h) - \ln(4)}{h} = \left( \frac{d}{dx} \ln x \right) \bigg|_{x = 4} = \frac{1}{x} \bigg|_{x = 4} = \frac{1}{4} \]

Answer B
8) The function \( f \) is defined by \( f(x) = \frac{x}{x + 2} \). What points \((x, y)\) on the graph of \( f \) have the property that the line tangent to \( f \) at \((x, y)\) has slope of \( \frac{1}{2} \)?

Recall that the slope of the tangent line of a function at a point is equal to the derivative of the function at that point. Then,

\[
\frac{d}{dx} \left( \frac{x}{x + 2} \right) = \frac{(x + 2) \cdot 1 - x \cdot 1}{(x + 2)^2} = \frac{x + 2 - x}{(x + 2)^2} = \frac{2}{(x + 2)^2}
\]

The problem tells us that this is equal to a slope of \( \frac{1}{2} \). So,

\[
\frac{2}{(x + 2)^2} = \frac{1}{2}
\]

\[
4 = x^2 + 4x + 4
\]

\[
0 = x^2 + 4x
\]

\[
x = \{0, -4\}
\]

To get the points, substitute the \( x \)-values into \( f(x) = \frac{x}{x + 2} \) to get the points: \((0, 0), (-4, 2)\)

**Answer C**

9) The line \( y = 5 \) is a horizontal asymptote to the graph of which of the following functions?

\[(A) \ y = \frac{\sin(5x)}{x} \quad (B) \ y = 5x \quad (C) \ y = \frac{1}{x - 5} \quad (D) \ y = \frac{5x}{1-x} \quad (E) \ y = \frac{20x^2 - x}{1 + 4x^2}\]

Horizontal asymptotes occur at the limits as \( x \to \infty \) and as \( x \to -\infty \).

Let’s try limits as \( x \to \infty \).

\[(A) \lim_{x \to \infty} \frac{\sin(5x)}{x} = 0 \]

\[(B) \lim_{x \to \infty} 5x = \infty \]

\[(C) \lim_{x \to \infty} \frac{1}{x - 5} = 0 \]

\[(D) \lim_{x \to \infty} \frac{5x}{1-x} = \lim_{x \to \infty} \frac{5}{-1} = -5 \quad \text{using L'Hospital's Rule} \]

\[(E) \lim_{x \to \infty} \frac{20x^2 - x}{1 + 4x^2} = \frac{40x - 1}{8x} = \frac{40}{8} = 5 \quad \text{using L'Hospital's Rule} \]

**Answer E**
10) If \( f(x) = 16\sqrt{x} \) then \( f''(4) \) is equal to

\[ f(x) = 16x^{1/2} \]
\[ f'(x) = 16 \cdot \left( \frac{1}{2}x^{-1/2} \right) = 8x^{-1/2} \]
\[ f''(x) = 8 \cdot \left( -\frac{1}{2}x^{-3/2} \right) = -4x^{-3/2} = \frac{-4}{x^{3/2}} \]
\[ f''(4) = \frac{-4}{4^{3/2}} = \frac{-4}{8} = -\frac{1}{2} \]

Answer A

11) If \((x + 2y) \cdot \frac{dy}{dx} = 2x - y\), what is the value of \( \frac{d^2y}{dx^2} \) at the point \((3, 0)\)?

\[ \frac{dy}{dx} = \frac{2x - y}{x + 2y} \]

Notice that, at the point \((3, 0)\),
\[ \frac{dy}{dx} = \frac{2 \cdot 3 - 0}{3 + 2 \cdot 0} = \frac{6}{3} = 2 \]
\[ \frac{d^2y}{dx^2} = \frac{(x + 2y) \cdot \frac{d}{dx}(2x - y) - (2x - y) \cdot \frac{d}{dx}(x + 2y)}{(x + 2y)^2} \]
\[ = \frac{(x + 2y) \cdot \left( 2 - \frac{dy}{dx} \right) - (2x - y) \cdot \left( 1 + 2 \frac{dy}{dx} \right)}{(x + 2y)^2} \]

Now, substitute in \( x = 3, \ y = 0 \), \( \frac{dy}{dx} = 2 \)
\[ \frac{d^2y}{dx^2} = \frac{(3 + 2 \cdot 0) \cdot (2 - 2) - (2 \cdot 3 - 0) \cdot (1 + 2 \cdot 2)}{(3 + 2 \cdot 0)^2} = \frac{3 \cdot 0 - 6 \cdot 5}{9} = -\frac{10}{3} \]

Answer A
12) \( \frac{d}{dx}[e^{\sin x}] = \) 

(A) \( e^{\sin x} \)  \hspace{1cm} (B) \( e^{\cos x} \)  \hspace{1cm} (C) \( \cos x \)  \hspace{1cm} (D) \( e^{\sin x} \cos x \)  \hspace{1cm} (E) \( e^{\cos x} \sin x \)

\[
\frac{d}{dx} e^{\sin x} = e^{\sin x} \cdot \frac{d}{dx} (\sin x) \]

\[= e^{\sin x} \cdot \cos x \]  \hspace{1cm} \text{Answer D}

13) If \( y = \frac{2-x}{3x+1} \), then \( \frac{dy}{dx} = \)

(A) \( -\frac{9}{(3x+1)^2} \)  \hspace{1cm} (B) \( -\frac{7}{(3x+1)^2} \)  \hspace{1cm} (C) \( \frac{6x-5}{(3x+1)^2} \)  \hspace{1cm} (D) \( \frac{7}{(3x+1)^2} \)  \hspace{1cm} (E) \( \frac{7-6x}{(3x+1)^2} \)

\[
y = \frac{2-x}{3x+1}
\]

\[
\frac{dy}{dx} = \frac{(3x+1) \cdot \frac{d}{dx} (2-x) - (2-x) \cdot \frac{d}{dx} (3x+1)}{(3x+1)^2}
\]

\[= \frac{(3x+1) \cdot (-1) - (2-x) \cdot (3)}{(3x+1)^2} = -\frac{3x-1 - 6x}{(3x+1)^2} = -\frac{7}{(3x+1)^2}
\]  \hspace{1cm} \text{Answer B}

Part II: Graphing Calculator is Allowed

14) Let \( f \) be a function that is continuous on the closed interval \([2, 4]\) with \( f(2) = 10 \) and \( f(4) = 20 \). Which of the following is guaranteed by the Intermediate Value Theorem?

(A) \( f(x) = 13 \) has at least one solution in the open interval \((2, 4)\). \text{TRUE} since 13 is between 10 and 20. One way to think about it is that to get from 10 to 20, the function must go through 13. \hspace{1cm} \text{Answer A}

B) \( f(3) = 15 \) \text{FALSE}. \( f(3) \) can be anything, including values below 10 or above 20.

C) \( f \) attains a maximum on the open interval \((2, 4)\). \text{FALSE}. \( f \) may have a maximum at \( x = 4 \). For example, consider the line connecting \((2, 10)\) and \((4, 20)\). It has a maximum at \( x = 4 \), not in the open interval \((2, 4)\).

D) \( f''(x) = 5 \) has at least one solution in the open interval \((2, 4)\). \text{TRUE}, but this is not required by the Intermediate Value Theorem, which deals with the values of the function, not the values of the derivative.

E) \( f''(x) > 0 \) for all \( x \) in the open interval \((2, 4)\). \text{FALSE}. The curve could slope downward in some sub-interval between 2 and 4.

**Intermediate Value Theorem**: If a function, \( f \), is continuous on the interval \([a, b]\), and \( d \) is a value between \( f(a) \) and \( f(b) \), then there is a value \( c \) in \([a, b]\) such that \( f(c) = d \).
15) Let:
\[ x = \text{the distance of the person from the streetlight} \]
\[ y = \text{the length of the shadow} \]

A person whose height is 6 feet is walking away from the base of a streetlight along a straight path at a rate of 4 feet per second. If the height of the streetlight is 15 feet, what is the rate at which the person's shadow is lengthening?

(A) 15 ft/sec  (B) 2.667 ft/sec  (C) 3.75 ft/sec  (D) 6 ft/sec  (E) 10 ft/sec

Using the above drawing, we are looking for \( \frac{dy}{dt} \) when \( \frac{dx}{dt} = 4 \) ft/sec.

Comparing the small rectangle to the large triangle, from Geometry:
\[
\frac{15}{6} = \frac{x + y}{y} \\
15y = 6x + 6y \\
y = \frac{2}{3}x \\
\frac{dy}{dx} = \frac{2}{3}
\]

Note that:
\[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \\
\frac{dy}{dt} = \frac{2}{3} \times 4 = \frac{8}{3} \\
\frac{dy}{dt} = 2.667 \text{ ft/sec}
\]

Answer B

16) A particle moves along the x-axis so that its position at any time \( t \) (in seconds) is \( s(t) = 2t^2 + 3t + 5 \). The acceleration of the object at \( t = 2 \) seconds is:

(A) 19 units/s²  (B) 11 units/s²  (C) 16 units/s²  (D) 4 units/s²  (E) 0 units/s²

Acceleration is the second derivative of the position curve (the first derivative is velocity).
\[
s(t) = 2t^2 + 3t + 5 \\
\frac{ds}{dt} = 4t + 3 \\
\frac{d^2s}{dt^2} = 4
\]

Note that the acceleration is the same for all values of \( t \). So,
\[
\left. \frac{d^2s}{dt^2} \right|_{t=2} = 4 \text{ units/sec}^2
\]

Answer B
Free Response: Part I: No Calculator

17) (1981 AB) Let \( f \) be defined by \( f(x) = \begin{cases} 2x + 1, & x \leq 2 \\ \frac{1}{2}x^2 + k, & x > 2 \end{cases} \).

(A) For what value(s) of \( k \) will \( f \) be continuous at \( x = 2 \)? Justify with the definition.

(B) Find the average rate of change of the function on the interval \([1, 4]\).

(C) Find the instantaneous rate of change of the function at \( x = 4 \).

(A) **Continuity:** A function, \( f \), is continuous at \( x = c \) if:

a. \( f(c) \) is defined,

b. \( \lim_{x \to c} f(x) \) exists, and

c. \( \lim_{x \to c} f(x) = f(c) \)

In the problem given, this boils down to: \( 2x + 1 = \frac{1}{2}x^2 + k \).

When \( x = 2 \):

\[
2x + 1 = \frac{1}{2}x^2 + k \quad \text{becomes:} \quad 5 = 2 + k \quad \Rightarrow \quad k = 3
\]

(B) The average rate of change is the slope of the line connecting the endpoints of the interval:

\[ x = 1: \quad f(1) = 2(1) + 1 = 3 \]

\[ x = 4: \quad f(4) = \frac{1}{2}(4)^2 + 3 = 11 \]

Slope \( = \frac{11 - 3}{4 - 1} = \frac{8}{3} \)

(C) The instantaneous rate of change at \( x = 4 \) is the slope of the curve at \( x = 4 \). For \( x > 2 \):

\[ y = \frac{1}{2}x^2 + 3 \]

\[
\frac{dy}{dx} = x \quad \text{then,} \quad \frac{dy}{dx} \bigg|_{x=4} = 4
\]
18) Let \( f \) be the function defined by the equation \( f(x) = x^3 + 2x^2 - 6x \).

(a) Find the equations of the lines tangent and normal to the graph at the point \((2, f(2))\).
(b) Find the equation(s) of the line(s) tangent to the graph of \( f \) and parallel to the line \( y = 7x + 3 \).
(c) At what value of \( x \), if any, is the tangent line horizontal?

\[

tangent \ line: \quad y - 4 = 14(x - 2), \quad \text{using point-slope form}
\]

\[

\text{normal \ line: \quad y - 4 = -\frac{1}{14}(x - 2), \quad \text{normal \ slope \ is \ opposite \ reciprocal \ of \ tangent \ slope.}}
\]
(B) Find the equation(s) of the line(s) tangent to the graph of \( f \) and parallel to the line \( y = 7x + 3 \).

We want the points where the slope of the curve is 7
(i.e., the slope of \( y = 7x + 3 \))

\[
f'(x) = 3x^2 + 4x - 6 = 7\]

\[3x^2 + 4x - 13 = 0\]

Use the quadratic formula to determine that:

\[x = \{ 1.519, -2.852 \}\]

so the points are: \((1.519, -0.993)\)
and \((-2.852, 10.179)\)

Note: to get the \( y \)-values of the points, substitute the \( x \)-values into the original equation:

\[f(x) = x^3 + 2x^2 - 6x\]

Tangent Equation 1 (point-slope form):

\[y + 0.993 = 7(x - 1.519)\]

Tangent Equation 2 (point-slope form):

\[y - 10.179 = 7(x + 2.852)\]

(C) At what value of \( x \), if any, is the tangent line horizontal?

The tangent line is horizontal when the derivative is zero.

\[f'(x) = 3x^2 + 4x - 6 = 0\]

Using the quadratic formula,

\[x = \frac{-4 \pm \sqrt{4^2 - 4(3)(-6)}}{2 \cdot 3} = \frac{-4 \pm \sqrt{88}}{6}\]

\[x = \frac{-2 \pm \sqrt{22}}{3} = \{ -2.230, 0.897 \}\]
19) The function \( f \) is defined by \( f(x) = \sqrt{25 - x^2} \) for \(-5 \leq x \leq 5\).

(a) Find \( f'(x) \).

(b) Write an equation for the line tangent to the graph of \( f \) at \( x = -3 \).

(c) Let \( g \) be the function defined by \( g(x) = \begin{cases} f(x) & \text{for } -5 \leq x \leq -3 \\ x + 7 & \text{for } -3 < x \leq 5 \end{cases} \)

Is \( g \) continuous at \( x = -3 \)? Use the definition of continuity to explain your answer.

(a) \( f(x) = (25 - x^2)^{1/2} \)

\[ f'(x) = \frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x) \]

\[ f'(x) = -\frac{x}{\sqrt{25 - x^2}} \quad x \neq \pm 5 \]

(b) \( f(-3) = \sqrt{25 - (-3)^2} = 4 \)

\[ f'(-3) = -\frac{-3}{\sqrt{25-(-3)^2}} = \frac{3}{4} \]

So, the tangent line has slope \( \frac{3}{4} \) and goes through the point \((-3, 4)\).

Tangent Equation is: \( y - 4 = \frac{3}{4}(x + 3) \) (using point-slope form)

(c) **Continuity:** A function, \( f \), is continuous at \( x = c \) iff:

a. \( f(c) \) is defined,

b. \( \lim_{x \to c} f(x) \) exists, and

c. \( \lim_{x \to c} f(x) = f(c) \)

\[ g(-3) = f(-3) = 4 \quad \text{(from above)} \]

\[ \lim_{x \to -3^-} \sqrt{25 - x^2} = 4 \quad \text{left and right limits} \]

\[ \lim_{x \to -3^+} x + 7 = 4 \]

Since the limits from the left and right are the same at \( x = -3 \) and are equal to the value of \( g(-3) \), which exists, we conclude that \( g \) is continuous at \( x = -3 \).