Taylor and Maclaurin Series

First Development of Taylor Series Formula

Let's derive a polynomial approximation for a given function, $f(x)$, of the form:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + \ \dots$$

When $x = a$, all but the constant term zero out and this equation simplifies to $f(a) = c_0$.

Taking the first derivative of $f(x)$, we get:

$$f'(x) = c_1 + 2 \cdot c_2(x-a) + 3 \cdot c_3(x-a)^2 + 4 \cdot c_4(x-a)^3 + 5 \cdot c_5(x-a)^4 + \ \dots$$

When $x = a$, this equation simplifies to $f'(a) = c_1$.

Taking the second derivative of $f(x)$, we get:

$$f''(x) = 2! \cdot c_2 + 3 \cdot 2 \cdot c_3(x-a) + 4 \cdot 3 \cdot c_4(x-a)^2 + 5 \cdot 4 \cdot c_5(x-a)^3 + \ \dots$$

When $x = a$, this equation simplifies to $f''(a) = 2! \cdot c_2$.

Taking the third derivative of $f(x)$, we get:

$$f'''(x) = 3! \cdot c_3 + 4 \cdot 3 \cdot 2 \cdot c_4(x-a) + 5 \cdot 4 \cdot 3 \cdot c_5(x-a)^2 + \ \dots$$

When $x = a$, this equation simplifies to $f'''(a) = 3! \cdot c_3$.

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Taking the n^{th} derivative of $f(x)$, we get:

$$f^{(n)}(x) = n! \cdot c_n + \frac{(n+1)!}{1!} \cdot c_{n+1}(x-a) + \frac{(n+2)!}{2!} \cdot c_{n+2}(x-a)^2 + \frac{(n+3)!}{3!} \cdot c_{n+3}(x-a)^3 + \ \dots$$

When $x = a$, this equation simplifies to $f^{(n)}(a) = n! \cdot c_n$.

Following the above pattern, each c-value can be calculated as: $c_k = \frac{f^{(k)}(a)}{k!}$.

Plugging the c-values into the original equation above, we get:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ \dots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Note that we define $f^{(0)}(x)$ to be the zero-th derivative of $f(x)$, i.e. $f(x)$ itself.

Second Development of Taylor Series Formula

To derive the Taylor Polynomial for a given function, $f(x)$, start with the function's $(n+1)^{st}$ derivative, $f^{(n+1)}(x)$. Integrate this derivative from a to x. Let's use t as the integration variable in order to avoid confusion. Also, **constant terms** will be shown in bold font.

$$\int_a^x f^{(n+1)}(t)\,dt = f^{(n)}(t)\,\Big|_a^x = f^{(n)}(x) - \boldsymbol{f^{(n)}(a)}$$

On the right side of the equal sign, $f^{(n)}(x)$ is the n^{th} derivative of $f(x)$, and $\boldsymbol{f^{(n)}(a)}$ is the n^{th} derivative of $f(x)$ evaluated at $x = a$, which is a constant.

Continue integrating this expression:

$$\int_a^x \left(\int_a^x f^{(n+1)}(t)\,dt\right) dt = \int_a^x \left(f^{(n)}(t) - \boldsymbol{f^{(n)}(a)}\right) dt = \left(f^{(n-1)}(t) - \boldsymbol{f^{(n)}(a)}\cdot t\right)\Big|_a^x$$

$$= \left[f^{(n-1)}(x) - \boldsymbol{f^{(n-1)}(a)}\right] - \left[\boldsymbol{f^{(n)}(a)}\cdot(x-a)\right]$$

$$\int_a^x \left(\int_a^x \left(\int_a^x f^{(n+1)}(t)\,dt\right) dt\right) dt = \int_a^x \left(\left[f^{(n-1)}(t) - \boldsymbol{f^{(n-1)}(a)}\right] - \left[\boldsymbol{f^{(n)}(a)}\cdot(x-a)\right]\right) dx$$

$$= \left(f^{(n-2)}(t) - \boldsymbol{f^{(n-1)}(a)}\cdot t\right)\Big|_a^x - \left[\frac{\boldsymbol{f^{(n)}(a)}}{\boldsymbol{2!}}(x-a)^2\right]$$

$$= \left[f^{(n-2)}(x) - \boldsymbol{f^{(n-2)}(a)}\right] - \left[\boldsymbol{f^{(n-1)}(a)}\cdot(x-a)\right] - \left[\frac{\boldsymbol{f^{(n)}(a)}}{\boldsymbol{2!}}\cdot(x-a)^2\right]$$

Until after $(n+1)$ integrations, we have:

$$\int_a^x \ldots\left(\int_a^x f^{(n+1)}(t)\,dt\right) \ldots dt = \left[f(x) - \boldsymbol{f(a)}\right] - \left[\boldsymbol{f'(a)}\cdot(x-a)\right] - \left[\frac{\boldsymbol{f''(a)}}{\boldsymbol{2!}}\cdot(x-a)^2\right]$$

$$-\left[\frac{\boldsymbol{f'''(a)}}{\boldsymbol{3!}}\cdot(x-a)^3\right] - \ \ldots\ - \left[\frac{\boldsymbol{f^{(n)}(a)}}{\boldsymbol{n!}}\cdot(x-a)^n\right]$$

Rearranging terms, we get:

$$f(x) = f(a) + \boldsymbol{f'(a)}\cdot(x-a) + \frac{\boldsymbol{f^{(n)}(a)}}{\boldsymbol{2!}}\cdot(x-a)^2 + \frac{\boldsymbol{f'''(a)}}{\boldsymbol{3!}}\cdot(x-a)^3 + \ \ldots\ + \frac{\boldsymbol{f^{(n)}(a)}}{\boldsymbol{n!}}\cdot(x-a)^n$$

$$-\left[\int_a^x \ldots\left(\int_a^x \boldsymbol{f^{(n+1)}(t)\,dt}\right) \ldots \boldsymbol{dt}\right].$$

Note: this term can be re-written as: $\dfrac{\boldsymbol{f^{(n+1)}(x^*)}}{\boldsymbol{(n+1)!}}\boldsymbol{(x-a)^{n+1}}$

The last term, called the **LaGrange Remainder** ($\boldsymbol{R_n}$), is the result of the $(n+1)$ integrations. So,

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f^{(n)}(a)}{2!}\cdot(x-a)^2 + \frac{f'''(a)}{3!}\cdot(x-a)^3 + \ \ldots\ + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n + \boldsymbol{R_n}$$

Taylor Series Formula

A Taylor series is an expansion of a function around a given value of x. Generally, it has the following form around the point $x = a$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Maclaurin Series Formula

A Maclaurin series is a Taylor Series around the value $x = 0$. Generally, it has the following form:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\,x^k = f(0) + \frac{f'(0)}{1!}\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

Notice that each term has three components:

- A function or derivative value at $x = a$: $\boldsymbol{f^{(k)}(a)}$
- A factorial in the denominator: $\boldsymbol{k!}$
- $(x-a)$ to some power: $\boldsymbol{(x-a)^k}$

So, we can construct a Taylor Series using a table like this:

$\boldsymbol{k}$	$\boldsymbol{f^{(k)}(x)}$	$\boldsymbol{f^{(k)}(a)}$	$\boldsymbol{k!}$	$\boldsymbol{(x-a)^k}$	$\boldsymbol{\frac{f^{(k)}(a)}{k!}(x-a)^k}$
0	$f(x)$	$f(a)$	1	1	$f(a)$
1	$f'(x)$	$f'(a)$	1	$(x-a)$	$f'(a)(x-a)$
2	$f''(x)$	$f''(a)$	2	$(x-a)^2$	$\frac{f''(a)}{2}(x-a)^2$
3	$f'''(x)$	$f'''(a)$	6	$(x-a)^3$	$\frac{f'''(a)}{6}(x-a)^3$
4	$f^{iv}(x)$	$f^{iv}(a)$	24	$(x-a)^4$	$\frac{f^{iv}(a)}{24}(x-a)^4$
Construct as many rows as are needed for the required task.					
Total					Sum of above

An example of how to use this table is provided on the next page.

Example:

Find the Maclaurin expansion for $f(x) = \ln(1+x)$ to six terms.

Note that in a Maclaurin expansion, we let $a = 0$, which simplifies the resulting expressions. Let's make a table for the expansion of $f(x) = \ln(1+x)$:

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$k!$	$(x-0)^k$	$\frac{f^{(k)}(a)}{k!}(x-0)^k$
0	$\ln(1+x)$	0	1	1	0 (not a term)
1	$\frac{1}{1+x}$	1	1	x	x
2	$-\frac{1}{(1+x)^2}$	-1	2	x^2	$-\frac{1}{2}x^2$
3	$\frac{2}{(1+x)^3}$	2	6	x^3	$\frac{1}{3}x^3$
4	$-\frac{6}{(1+x)^4}$	-6	24	x^4	$-\frac{1}{4}x^4$
5	$\frac{24}{(1+x)^5}$	24	120	x^5	$\frac{1}{5}x^5$
6	$-\frac{120}{(1+x)^6}$	-120	720	x^6	$-\frac{1}{6}x^6$
Total	$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6$				

The answer to the stated problem is:

$$\ln(1+x) \sim x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}$$

In we were to continue the process and show the complete Maclaurin expansion, it would be:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, x^n}{n}$$

LaGrange Remainder

The form for a Taylor Series about $x = a$ that includes an error term is:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n)!}(x-a)^n + R_n(x)$$

The term $R_n(x)$ is called the Lagrange Remainder, and has the form:

$$R_n(x) = \frac{f^{(n+1)}(x^*)}{(n+1)!}(x-a)^{n+1}$$

where, x^* produces the greatest value of $f^{(n+1)}(x)$ between a and x.

This form is typically used to approximate the value of a series to a desired level of accuracy.

Example: Approximate $\sqrt{e}$ using five terms of the Maclaurin Series (i.e., the Taylor Series about $x = 0$) for e^x and estimate the maximum error in the estimate.

Using five terms and letting $x = \frac{1}{2}$, we get:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + R_4(x)$$

$$e^{1/2} \sim 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \frac{\left(\frac{1}{2}\right)^4}{4!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} = \mathbf{1.6484375}$$

To find the maximum potential error in this estimate, calculate:

$R_4(x) = \frac{f^{(5)}(x^*)}{5!}x^5$ for $x = \frac{1}{2}$ and x^* between 0 and $\frac{1}{2}$.

Since $f(x) = e^x$, the fifth derivative of f is: $f^{(5)}(x) = e^x$. The maximum value of this between $x = 0$ and $x = \frac{1}{2}$ occurs at $x = \frac{1}{2}$. Then,

$f^{(5)}\left(\frac{1}{2}\right) = e^{1/2} < 1.65$ based on our estimate of 1.6484375 above (we will check this after completing our estimate of the maximum error). Combining all of this,

$$R_4\left(\frac{1}{2}\right) = \frac{f^{(5)}\left(\frac{1}{2}\right)}{5!}\left(\frac{1}{2}\right)^5 < \frac{1.65}{5!}\left(\frac{1}{2}\right)^5 = \mathbf{0.0004297}$$

Note that the maximum value of $\sqrt{e}$, then, is $1.6484375 + 0.0004297 = 1.6488672$, which is less than the 1.65 used in calculating $R_4\left(\frac{1}{2}\right)$, so our estimate is good. The actual value of $\sqrt{e}$ is $1.6487212\ldots$.