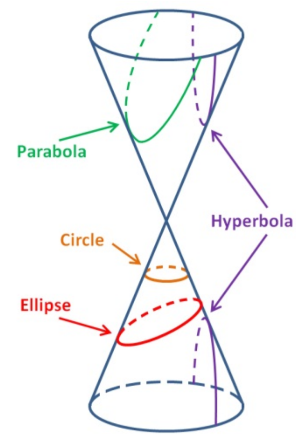
Algebra

Introduction to Conic Sections

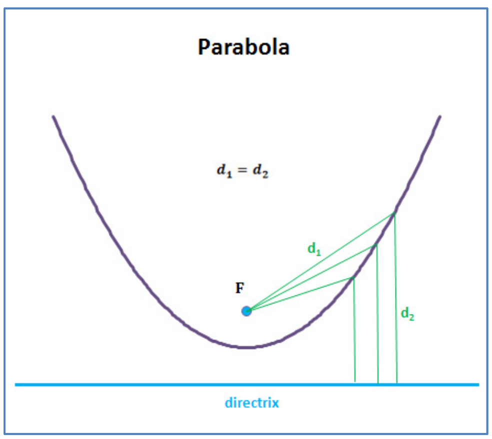
The intersection of a cone and a plane is called a **conic section**. There are four types of curves that result from these intersections that are of particular interest:

- Parabola
- Circle
- Ellipse
- Hyperbola

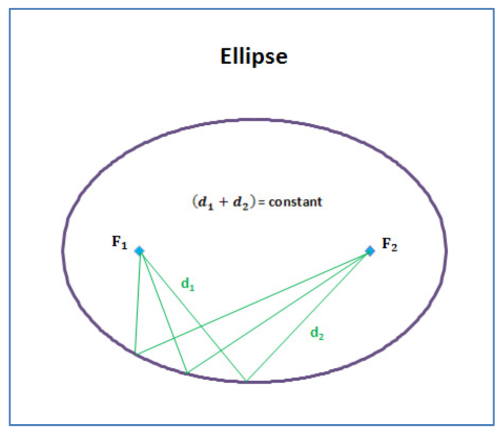
Each of these has a geometric definition, from which the algebraic form is derived.



Geometric Definitions

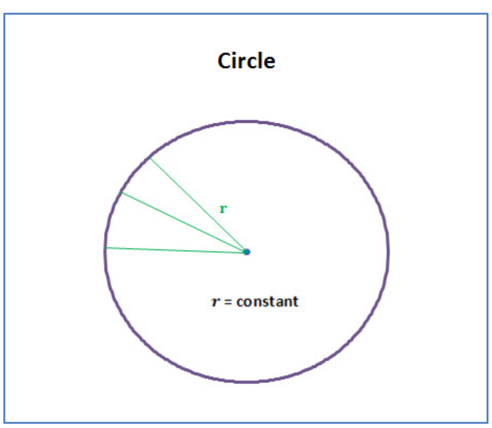


Parabola – The set of all points that are the same distance from a point (called the focus) and a line (called the Directrix).

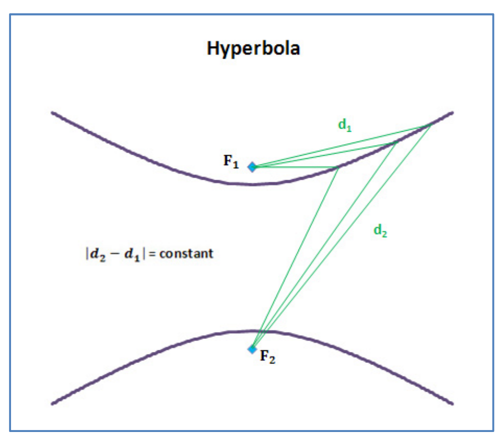


Ellipse – The set of all points for which the sum of the distances to two points (called foci) is constant.

Circle – The set of all points that are the same distance from a point (called the center). The distance is called the radius.



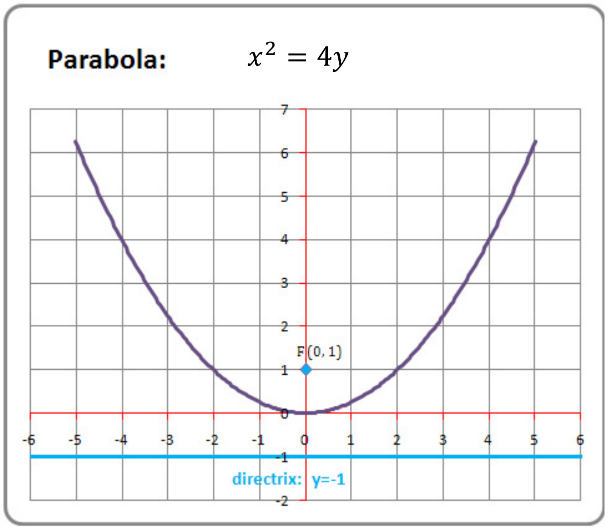
Hyperbola – The set of all points for which the difference of the distances to two points (called foci) is constant.



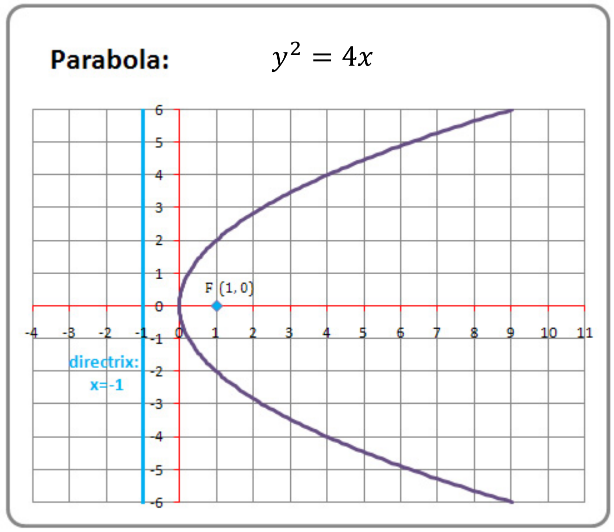
Algebra

Parabola with Vertex at the Origin (Standard Position)

Horizontal Directrix



Vertical Directrix



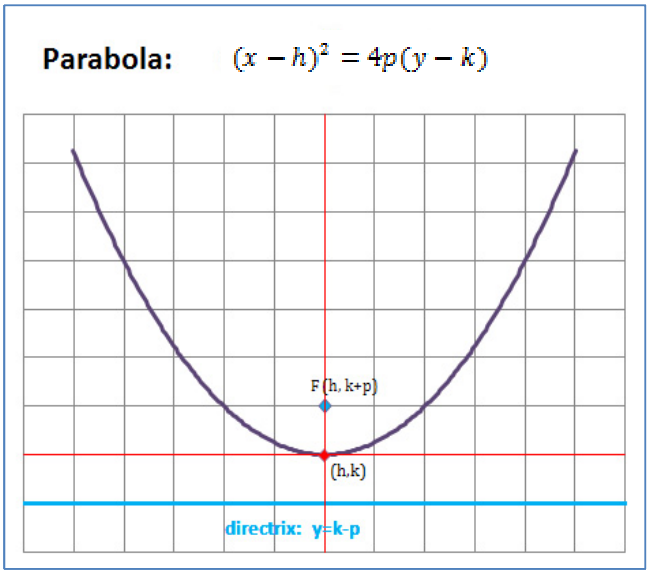
Characteristics of a Parabola in Standard Position		
	Horizontal Directrix	**Vertical Directrix**
Equation	$x^2 = 4py$	$y^2 = 4px$
If $p > 0$	opens up	opens right
If $p < 0$	opens down	opens left
Eccentricity ("e")	$e = 1$	$e = 1$
Value of p (in illustration)	$p = 1$	$p = 1$
Vertex	$(0, 0)$ - the origin	$(0, 0)$ - the origin
Focus	$(0, p)$	$(p, 0)$
Directrix	$y = -p$	$x = -p$
Axis of symmetry	$x = 0$ (y-axis)	$y = 0$ (x-axis)

Algebra

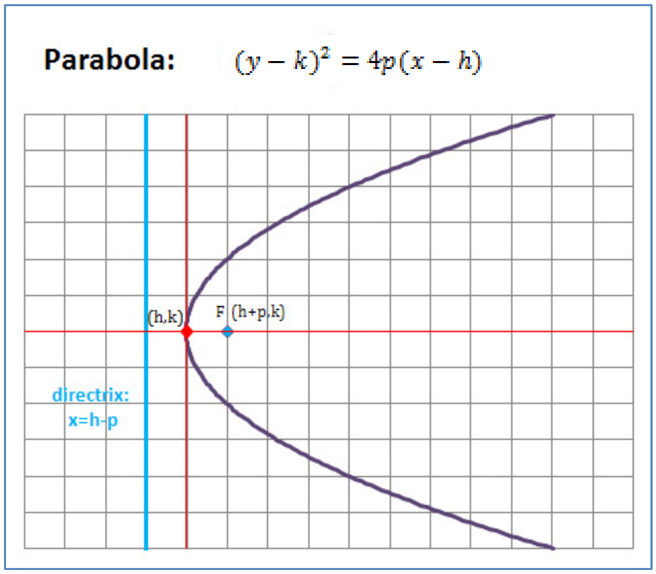
Parabola with Vertex at (h, k)

Horizontal Directrix	*Vertical Directrix*

Horizontal Directrix



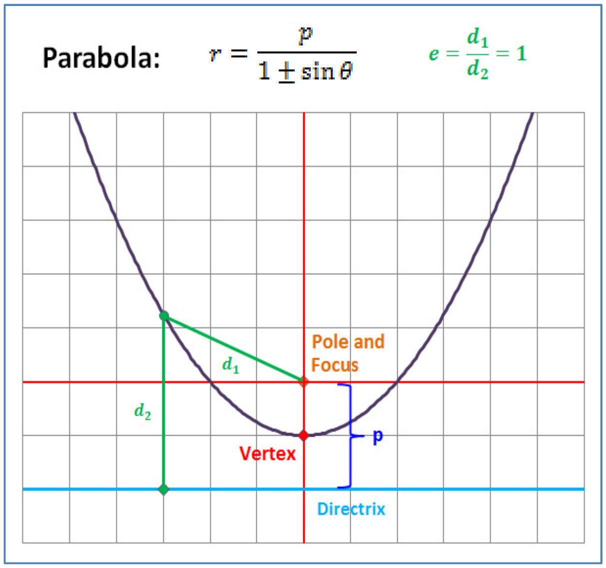
Vertical Directrix



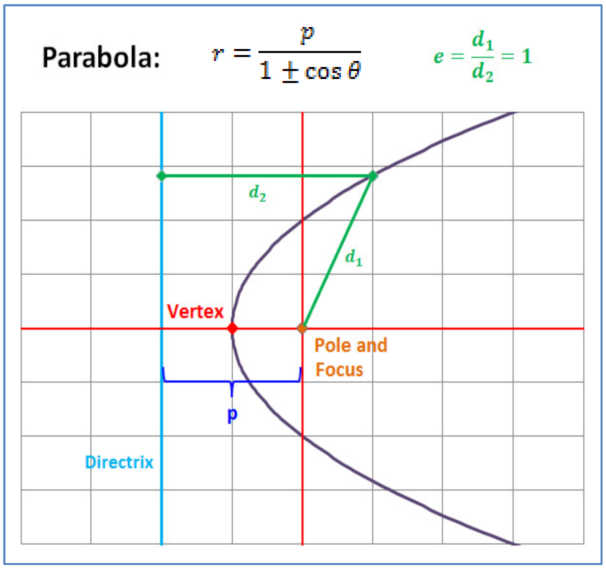
Characteristics of a Parabola with Vertex at Point (h, k)		
	Horizontal Directrix	**Vertical Directrix**
Equation	$(x-h)^2 = 4p(y-k)$	$(y-k)^2 = 4p(x-h)$
If $p > 0$	opens up	opens right
If $p < 0$	opens down	opens left
Eccentricity ("e")	$e = 1$	$e = 1$
Vertex	(h, k)	(h, k)
Focus	$(h, k+p)$	$(h+p, k)$
Directrix	$y = k-p$	$x = h-p$
Axis of symmetry	$x = h$	$y = k$

Algebra
Parabola in Polar Form

Horizontal Directrix

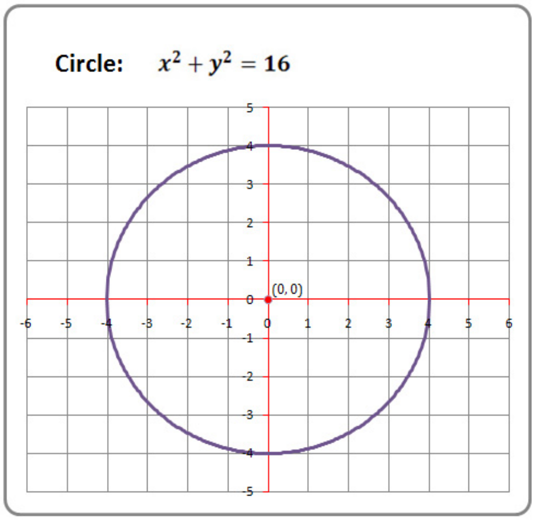


Vertical Directrix

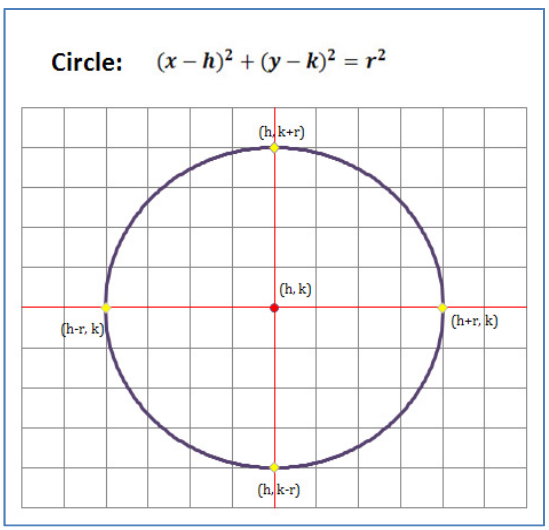


Characteristics of a Parabolas in Polar Form		
	Horizontal Directrix	**Vertical Directrix**
Equation (simplified)	$r = \frac{p}{1 \pm \sin\theta}$	$r = \frac{p}{1 \pm \cos\theta}$
If " $-$ " in denominator	opens up Directrix below Pole	opens right Directrix left of Pole
If " $+$ " in denominator	opens down Directrix above Pole	opens left Directrix right of Pole
Eccentricity ("e")	$e = 1$	$e = 1$
Focal Parameter ("p")	$p =$ distance between the Directrix and the Focus *Note: "p" in Polar Form is different from "p" in Cartesian Form*	
Coordinates of Key Points: *(change all instances of "–p" below to "p" if "+" is in the denominator)*		
Vertex	$(0, -p/2)$	$(-p/2, 0)$
Focus	(0,0)	(0,0)
Directrix	$y = -p$	$x = -p$

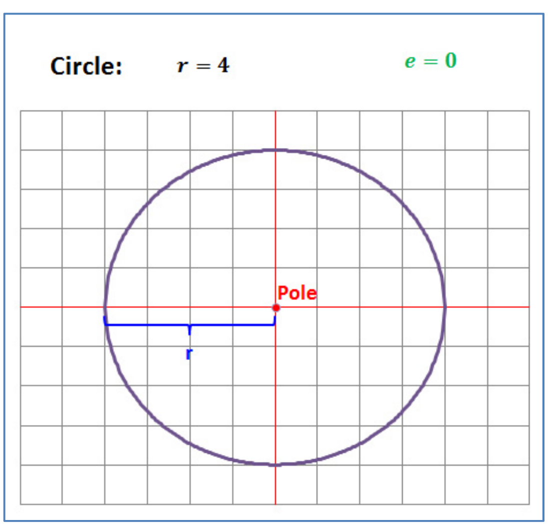
Algebra
Circles



Characteristics of a Circle in Standard Position	
Equation	$x^2 + y^2 = r^2$
Center	(0,0) - the origin
Radius	r
In the example	$r = 4$



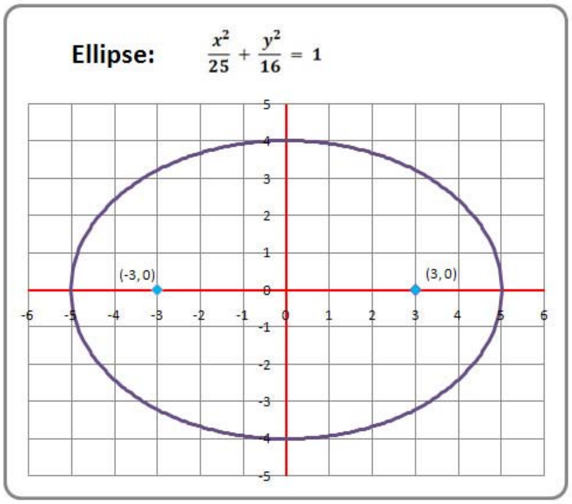
Characteristics of a Circle Centered at Point (h, k)	
Equation	$(x-h)^2 + (y-k)^2 = r^2$
Center	(h, k)
Radius	r



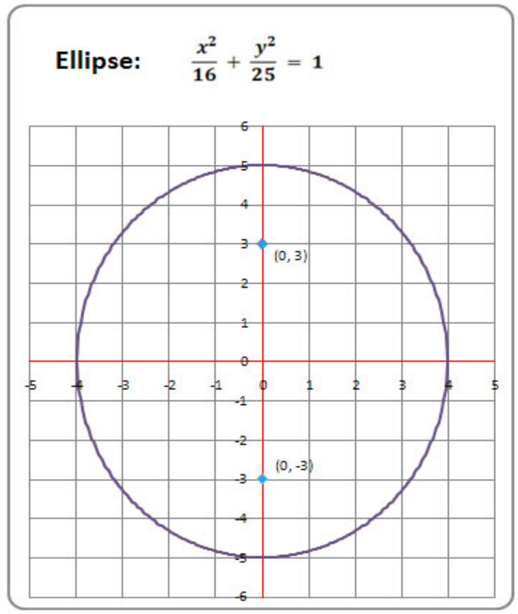
Characteristics of a Circle in Polar Form	
Equation	$r = constant$
Pole	$(0, 0)$
Radius	r

Algebra
Ellipse Centered on the Origin (Standard Position)

Horizontal Major Axis



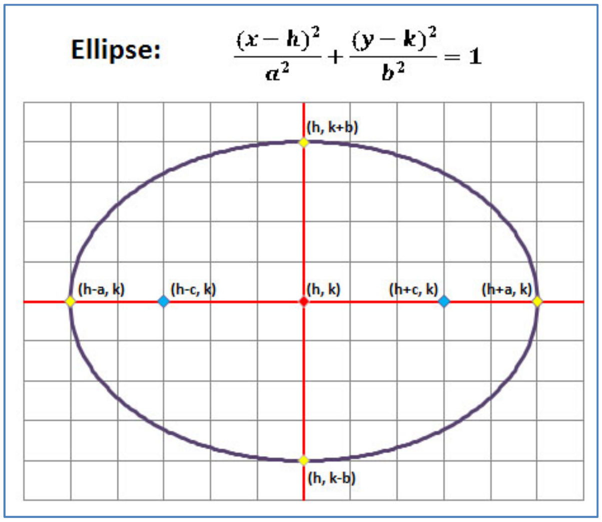
Vertical Major Axis



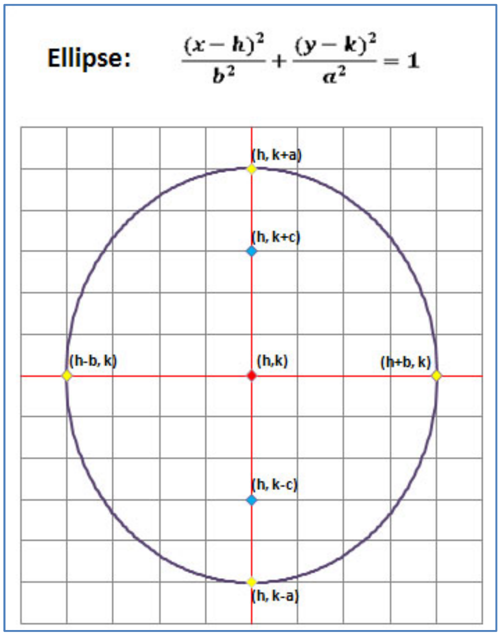
Characteristics of an Ellipse in Standard Position		
	Horizontal Major Axis	**Vertical Major Axis**
In the above example	$a = 5, \ b = 4, \ c = 3$	$a = 5, \ b = 4, \ c = 3$
Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$
Values of "a" and "b"	$a > b$	
Value of "c"	$c^2 = a^2 - b^2$	
Eccentricity ("e")	$e = c/a \qquad 0 < e < 1$	
Center	$(0,0)$ - the origin	
Major Axis Vertices	$(\pm a, 0)$	$(0, \pm a)$
Minor Axis Vertices	$(0, \pm b)$	$(\pm b, 0)$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Directrixes (not shown)	$x = \pm a/e$	$y = \pm a/e$

Algebra
Ellipse Centered at (h, k)

Horizontal Major Axis



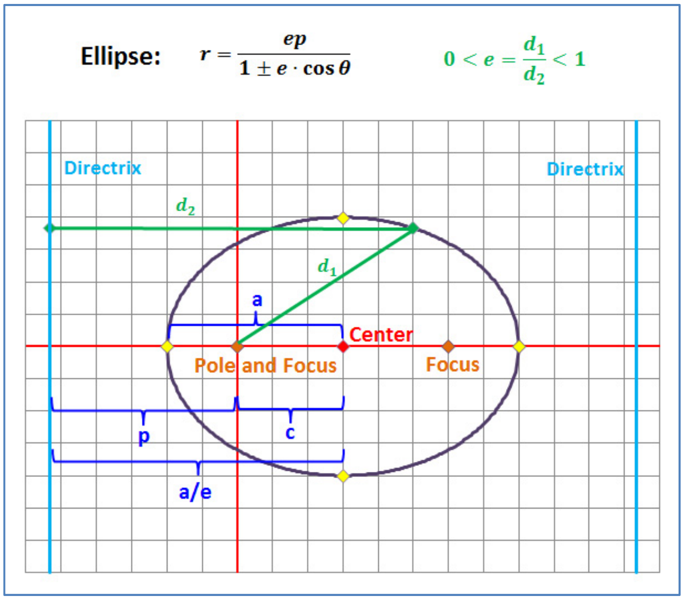
Vertical Major Axis



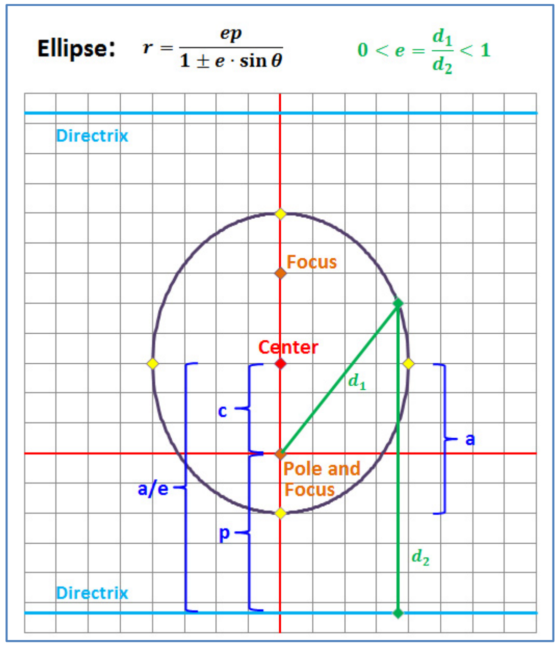
Characteristics of an Ellipse Centered at Point (h, k)		
	Horizontal Major Axis	**Vertical Major Axis**
Equation	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$
Values of "a" and "b"	$a > b$	
Value of "c"	$c^2 = a^2 - b^2$	
Eccentricity ("e")	$e = c/a \qquad 0 < e < 1$	
Center	(h, k)	
Major Axis Vertices	$(h \pm a,\ \ k)$	$(h,\ \ k \pm a)$
Minor Axis Vertices	$(h,\ \ k \pm b)$	$(h \pm b,\ \ k)$
Foci	$(h \pm c,\ \ k)$	$(h,\ \ k \pm c)$
Directrixes (not shown)	$x = h \pm a/e$	$y = k \pm a/e$

Algebra
Ellipse in Polar Form (Pole = One Focus)

Horizontal Major Axis



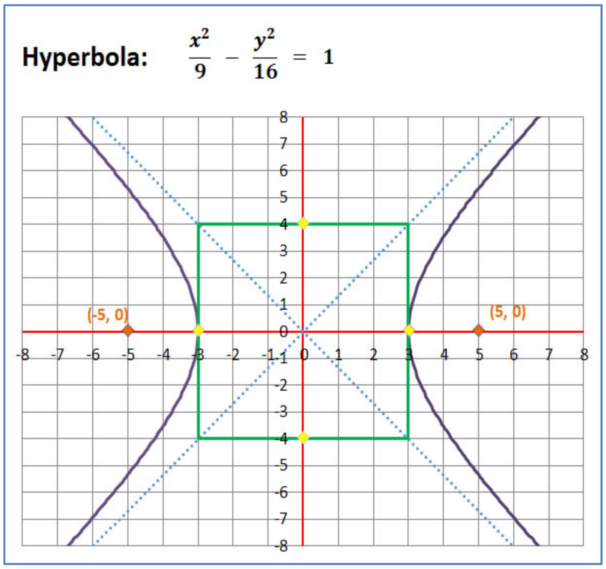
Vertical Major Axis



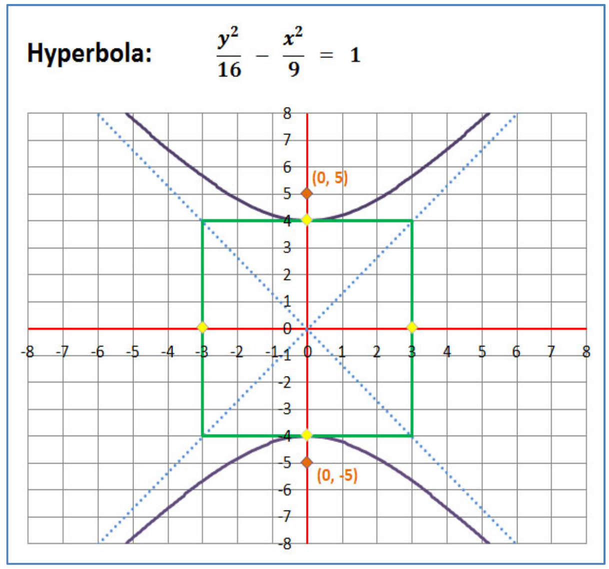
Characteristics of an Ellipse in Polar Form		
	Horizontal Major Axis	**Vertical Major Axis**
Equation	$r = \dfrac{ep}{1 \pm e \cdot \cos\theta}$	$r = \dfrac{ep}{1 \pm e \cdot \sin\theta}$
Value of "a"	a = distance from the Center to each major axis Vertex	
Value of "c"	c = distance from the Center to each Focus	
Eccentricity ("e")	$e = c/a \qquad 0 < e < 1$	
Focal Parameter ("p")	p = distance from each Focus to its Directrix $= a/e - c$	
Coordinates of Key Points:		
If " $-$ " in denominator	all coordinate values are shown below	
If " $+$ " in denominator	change all instances of "c", below, to "$-c$"	
Center	$(c, 0)$	$(0, c)$
Major Axis Vertices	$(c \pm a, 0)$	$(0, c \pm a)$
Foci	$(c \pm c, 0)$	$(0, c \pm c)$
Directrixes	$x = c \pm a/e$	$y = c \pm a/e$

Algebra
Hyperbola Centered on the Origin (Standard Position)

Horizontal Transverse Axis



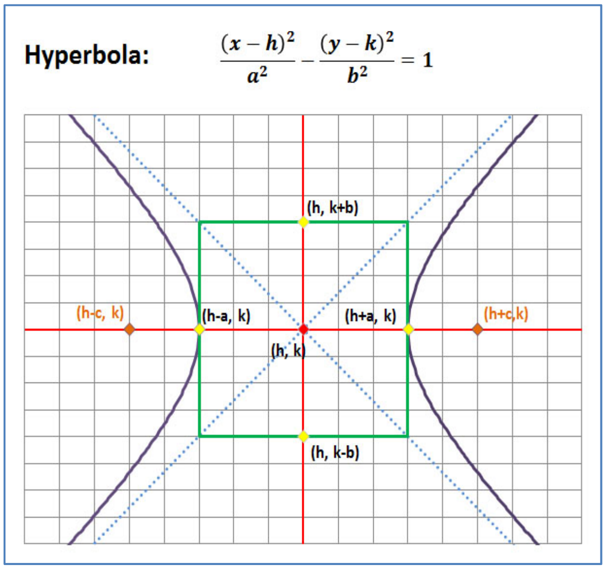
Vertical Transverse Axis



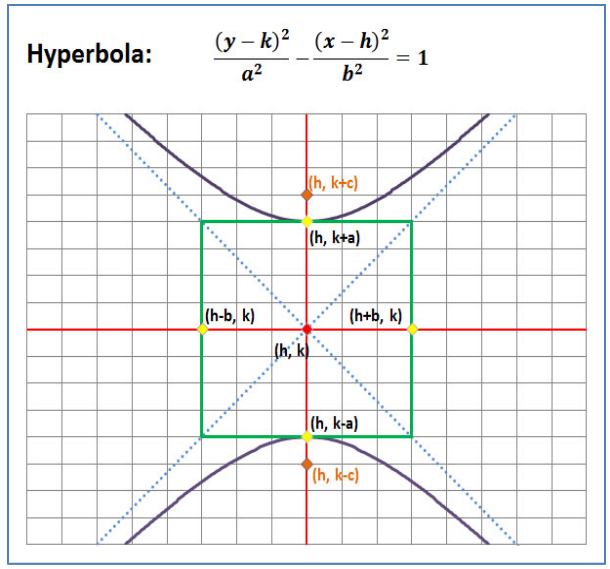
Characteristics of a Hyperbola in Standard Position		
	Horizontal Transverse Axis	**Vertical Transverse Axis**
In the above example	$a = 3, \; b = 4, \; c = 5$	$a = 4, \; b = 3, \; c = 5$
Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Value of "c"	$c^2 = a^2 + b^2$	
Eccentricity ("e")	$e = c/a \qquad e > 1$	
Center	(0,0) - the origin	
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Asymptotes	$y = \pm \frac{b}{a} x$	$y = \pm \frac{a}{b} x$
Directrixes (not shown)	$x = \pm a/e$	$y = \pm a/e$

Algebra
Hyperbola Centered at (h, k)

Horizontal Transverse Axis



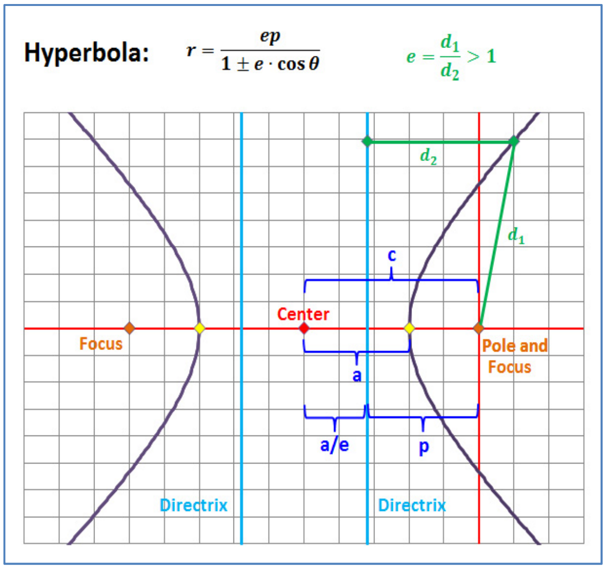
Vertical Transverse Axis



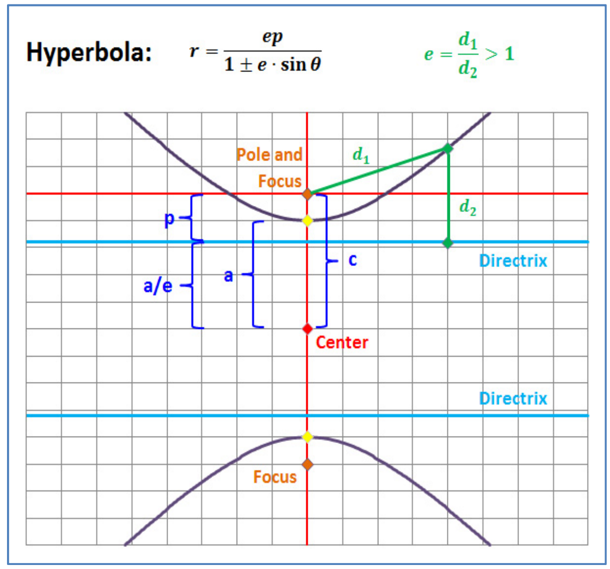
Characteristics of a Hyperbola Centered at Point (h, k)		
	Horizontal Transverse Axis	**Vertical Transverse Axis**
Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Value of "c"	$c^2 = a^2 + b^2$	
Eccentricity ("e")	$e = c/a \qquad e > 1$	
Center	(h, k)	
Vertices	$(h \pm a, \ k)$	$(h, \ k \pm a)$
Foci	$(h \pm c, \ k)$	$(h, \ k \pm c)$
Asymptotes	$(y-k) = \pm \frac{b}{a}(x-h)$	$(y-k) = \pm \frac{a}{b}(x-h)$
Directrixes (not shown)	$x = h \pm a/e$	$y = k \pm a/e$

Algebra
Hyperbola in Polar Form (Pole = One Focus)

Horizontal Transverse Axis



Vertical Transverse Axis



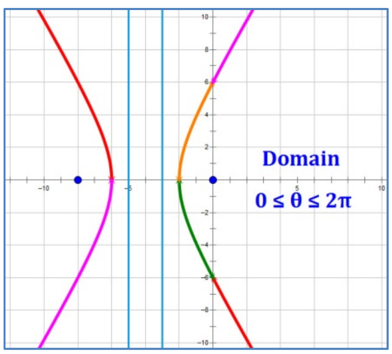
Characteristics of a Hyperbola in Polar Form		
	Horizontal Transverse Axis	**Vertical Transverse Axis**
Equation	$r = \dfrac{ep}{1 \pm e \cdot \cos\theta}$	$r = \dfrac{ep}{1 \pm e \cdot \sin\theta}$
Value of "a"	a = distance from the Center to each Vertex	
Value of "c"	c = distance from the Center to each Focus	
Eccentricity ("e")	$e = c/a \qquad e > 1$	
Focal Parameter ("p")	p = distance from each Focus to its Directrix $= c - a/e$	
Coordinates of Key Points:		
If " $-$ " in denominator	all coordinate values are shown below	
If " $+$ " in denominator	change all instances of "$-c$", below, to "c"	
Center	$(-c, 0)$	$(0, -c)$
Vertices	$(-c \pm a, 0)$	$(0, -c \pm a)$
Foci	$(-c \pm c, 0)$	$(0, -c \pm c)$
Directrixes	$x = -c \pm a/e$	$y = -c \pm a/e$

Algebra
Hyperbola in Polar Form (Pole = One Focus)
Partial Construction Over the Domain: 0 to 2π

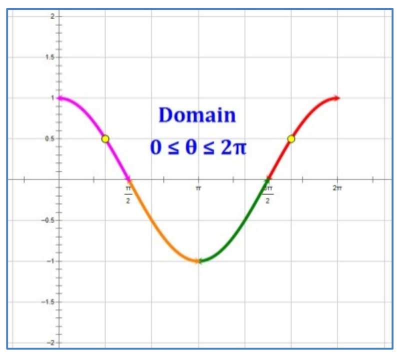
It is instructive to look at partial constructions of a hyperbola in polar form. Let's take a look at a curve constructed by varying θ from 0 to 2π, quadrant by quadrant:

$$\textbf{curve}: \quad r = \frac{6}{1 - 2\cos\theta} \qquad \textbf{parameters}: \quad a = 2, \quad c = 4, \quad e = 2, \quad p = 3$$

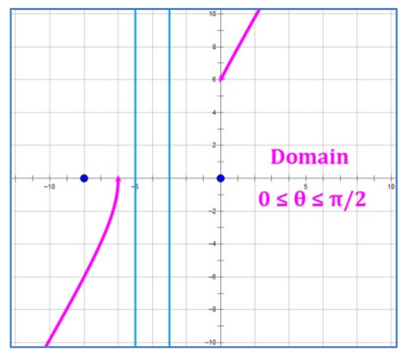
In the plots below, each quadrant in the domain is represented by a separate color. The portion of the curve added in each illustration is presented as a thicker line than the rest of the curve. The Foci of the curve are dark blue points and the Directrixes are light blue vertical lines.



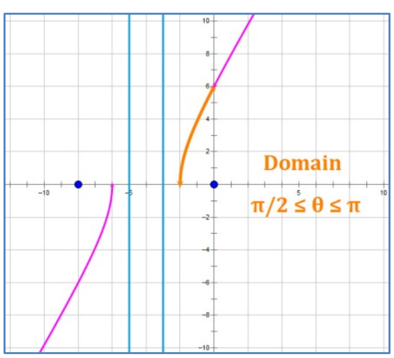
The **final curve** looks like this. The curve is plotted over the domain $0 \le \theta \le 2\pi$ but could also be plotted over the domain $-\pi \le \theta \le \pi$.



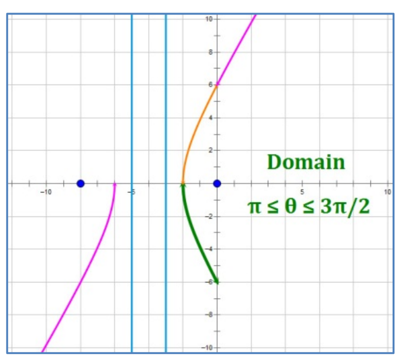
The **cosine function** has a major impact on how the curve graphs. Note the two yellow points where $\cos\theta = 0.5$. At these points, the curve is undefined.



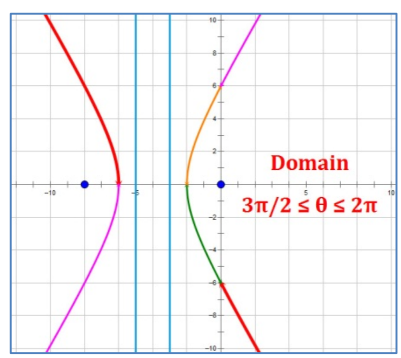
Q I: Domain $0 \le \theta \le \pi/2$. Note that the curve starts out on the left and switches to the right at $\theta = \pi/3$, where the curve is undefined.



Q II: Domain $\pi/2 \le \theta \le \pi$. The curve continues on the right side of the graph and gently curves down to the x-axis.



Q III: Domain $\pi \le \theta \le 3\pi/2$. The curve continues its gentle swing below the x-axis. Q III is essentially a reflection of the curve in Q II over the x-axis.



Q IV: Domain $3\pi/2 \le \theta \le 2\pi$. The curve continues on the right and switches to the left at $\theta = 5\pi/3$, where the curve is undefined.

Algebra
General Conic Equation – Classification

The General Case of the Conic Equation is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The second term may be omitted if the curve is not rotated relative to the axes in the Cartesian Plane, giving the simpler form:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

Conic Classification Tree

In this form, it is relatively easy to identify which type of curve the equation represents, using the following decision tree:

	yes	**Examples:**
Are Ax^2 and Cy^2 both missing? $(A = C = 0)$	The equation is not a conic. It is a **line**.	$3x - 2y - 4 = 0$ $-2x + 7y + 1 = 0$
no ↓ Is either Ax^2 or Cy^2 missing? $(A \cdot C = 0)$	The equation is a **parabola**.	$3y^2 - 2x + 7y - 3 = 0$ $x^2 + 4x + 2y - 1 = 0$
no ↓ Are the signs on Ax^2 and Cy^2 different? $(A \cdot C < 0)$	The equation is a **hyperbola**.	$4x^2 - 9y^2 - 8x + 27y + 2 = 0$ $-3x^2 + y^2 + 6x - 12y - 15 = 0$
no ↓ Are A and C the same number? $(A = C)$	The equation is a **circle**.	$x^2 + y^2 + 4x - 10y = 0$ $-4x^2 - 4y^2 + 16x - 8y + 5 = 0$
no ↓ Are A and C different numbers? $(A \neq C)$	The equation is an **ellipse**.	$5x^2 + 3y^2 + 20x - 12y - 3 = 0$ $-x^2 - 2y^2 + 8x + 12y + 13 = 0$

Algebra
General Conic Equation – Manipulation

After a conic equation is classified, it must be algebraically manipulated into the proper form. The steps involved are:

1. If there are negative coefficients in front of the square terms (Ax^2 and/or Cy^2), you may choose to eliminate them by multiplying the entire equation by -1.
2. Group the $\mathbf{x}$-terms on the left, the $\mathbf{y}$-terms on the right, and move the constant to the right side of the **=** sign. Set up parentheses around the $\mathbf{x}$-terms and the $\mathbf{y}$-terms.
3. Factor out the coefficients of the x^2 and y^2 terms.
4. Complete the squares for both the $\mathbf{x}$-terms and the $\mathbf{y}$-terms. Be careful to add the same numbers to both the right and left sides of the equations.
5. Reduce the completed squares to squared-binomial form.
6. If necessary, divide both sides by the required scalar and rearrange terms to obtain the proper form.

Example 1:

Solve: Equation	$-3x^2 + y^2 + 6x - 12y - 15 = 0$
Step 1: Change signs	$+3x^2 - y^2 - 6x + 12y + 15 = 0$
Step 2: Group variables	$(3x^2 - 6x + \_\_) - (y^2 - 12y + \_\_) = -15$
Step 3: Factor coefficients	$3(x^2 - 2x + \_\_) - (y^2 - 12y + \_\_) = -15$
Step 4: Complete Squares	$3(x^2 - 2x + 1) - (y^2 - 12y + 36) = -15 + 3 - 36$
Step 5: Reduce Square Terms	$3(x-1)^2 - (y-6)^2 = -48$
Step 6: Divide by (-48)	$-\frac{(x-1)^2}{16} + \frac{(y-6)^2}{48} = 1$
Rearrange Terms	$\frac{(y-6)^2}{48} - \frac{(x-1)^2}{16} = 1$

The final result is a hyperbola with center (1, 6) and a vertical transverse axis.

Example 2:

Solve: Equation	$-4x^2 - 4y^2 + 16x - 8y + 5 = 0$
Step 1: Change signs	$+4x^2 + 4y^2 - 16x + 8y - 5 = 0$
Step 2: Group variables	$(4x^2 - 16x + \_\_) + (4y^2 + 8y + \_\_) = 5$
Step 3: Factor Coefficients	$4(x^2 - 4x + \_\_) + 4(y^2 + 2y + \_\_) = 5$
Step 4: Complete Squares	$4(x^2 - 4x + 4) + 4(y^2 + 2y + 1) = 5 + 16 + 4$
Step 5: Reduce Square Terms	$4(x-2)^2 + 4(y+1)^2 = 25$
Step 6: Divide by 4	$(x-2)^2 + (y+1)^2 = \frac{25}{4}$

The final result is a circle with center (2, -1) and radius $\frac{5}{2}$.

ADVANCED

Algebra
Parametric Equations of Conic Sections

Parabola

Parametric Equations Centered at the Origin	Parametric Equations Centered at (h, k)
$x = 2pt$ $y = pt^2$	$x = 2pt + h$ $y = pt^2 + k$

Circle

Parametric Equations Centered at the Origin	Parametric Equations Centered at (h, k)
$x = r \cdot cos(t)$ $y = r \cdot sin(t)$	$x = r \cdot cos(t) + h$ $y = r \cdot sin(t) + k$

Ellipse

Parametric Equations Centered at the Origin	Parametric Equations Centered at (h, k)
$x = a \cdot cos(t)$ $y = b \cdot sin(t)$	$x = a \cdot cos(t) + h$ $y = b \cdot sin(t) + k$

Hyperbola

Parametric Equations Centered at the Origin	Parametric Equations Centered at (h, k)
$x = a \cdot sec(t)$ $y = b \cdot tan(t)$	$x = a \cdot sec(t) + h$ $y = b \cdot tan(t) + k$