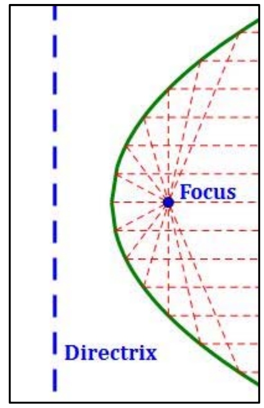
Reflection Property of a Parabola

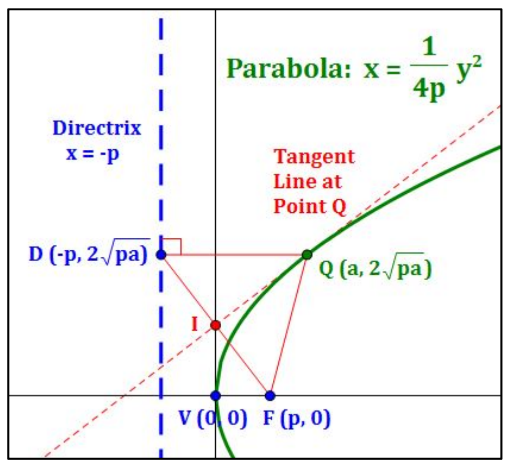


Reflection Property: Lines perpendicular to the directrix of a parabola reflect off the surface of the parabola and meet at its focus. The diagram at right illustrates this.

In the diagram, the red lines approach the parabola from the right, reflect off its surface and meet at the focus. This is how satellite dishes and parabolic antennas work. A paraboloid (a 3D version of the parabola) provides a wide area from which to collect signals. The signals reflect off the paraboloid surface and meet at a receiver placed at its focus, thus significantly amplifying the signal.

Similarly, the red lines could begin at the focus, radiate outward, reflect off the parabola to produce a wider signal. This is how headlights work. If you place a light source at the focus of a paraboloid, it will radiate outward in all directions, reflect off the paraboloid and produce a wide beam of light.

Proof: Let's prove the reflection property of a parabola.



Part 1: We will use coordinate geometry for the first portion of the proof. We are free to place the parabola anywhere in the coordinate plane, so we set its vertex at the Origin and orient it to the right. This results in a parabola of the general form: $x = \frac{1}{4p} y^2$. The diagram at right provides coordinates for key points relating to this parabola.

For a parabola of this form, the focus is p units to the right of the vertex and the Directrix is p units to the left of the vertex.

We construct a triangle (in red) with the following vertices: F, the focus; Q, a point anywhere on the parabola; and D, the point on the Directrix that makes $\overline{DQ}$ perpendicular to the Directrix. We also draw the line tangent to the parabola at Q, and define the intersection of this tangent and $\overline{DF}$ to be point I. (See the diagram.)

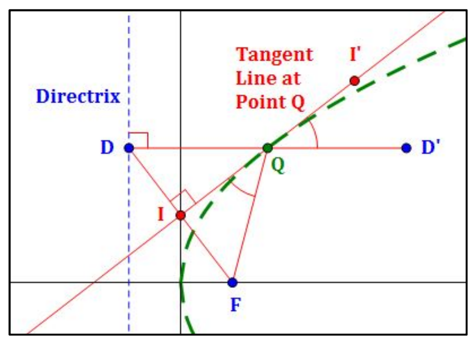
We want to show that $\overline{DF}$ and $\overline{QI}$ are perpendicular and, therefore, both $\angle QIF$ and $\angle QID$ are right angles. From basic Algebra, we know this will be true if the product of the slopes of $\overline{DF}$ and $\overline{QI}$ is -1. From the information in the above diagram, The slope of $\overline{DF}$ is $m_{\overline{DF}} = \frac{2\sqrt{pa}-0}{-p-p} = \frac{2\sqrt{pa}}{-2p} = -\sqrt{\frac{a}{p}}$

To find the slope of $\overline{QI}$, we take the derivative of the equation of the parabola at point Q. Since $x = \frac{1}{4p}y^2$, we have $y = 2\sqrt{px}$, and so $\frac{dy}{dx} = \frac{2\sqrt{p}}{2\sqrt{x}} = \sqrt{\frac{p}{x}}$. At point Q, $x = a$, so we have $\frac{dy}{dx} = \sqrt{\frac{p}{x}} = \sqrt{\frac{p}{a}}$, so the slope of $\overline{QI}$ is $m_{\overline{QI}} = \sqrt{\frac{p}{a}}$

Finally, $m_{\overline{DF}} \cdot m_{\overline{QI}} = -\sqrt{\frac{a}{p}} \cdot \sqrt{\frac{p}{a}} = -1$. We conclude that $\overline{DF}$ and $\overline{QI}$ are perpendicular, and therefore, both $\angle QIF$ and $\angle QID$ are right angles. *QED*

Part 2: Given the above result, we will use regular Geometry to prove the Reflection Property of a Parabola, i.e., that lines perpendicular to the directrix of a parabola reflect off the surface of the parabola and meet at its focus.

Consider the diagram at right. It is similar to the diagram above, but we have removed the coordinates and added points I' (on $\overleftrightarrow{IQ}$) and D' (on $\overleftrightarrow{DQ}$) so we can discuss the necessary angles.



For a line reflecting off a surface, the angle of reflection is equal to the angle of incidence. In the illustration, $\angle IQF$ and $\angle I'QD'$ are the angles of reflection and incidence. So we need only prove that $\angle IQF \cong \angle I'QD'$ in order to prove the Reflection Property.

From the diagram, we have the following:

- $\angle QIF$ and $\angle QID$ are right angles (from the first part of this proof, above). Therefore, ΔQIF and ΔQID are right triangles.
- $\overline{QF} \cong \overline{QD}$ by the definition of a parabola. Further, $\overline{QF}$ is the hypotenuse of ΔQIF, and $\overline{QD}$ is the hypotenuse of ΔQID.
- $\overline{QI} \cong \overline{QI}$ by the reflexive property of congruence. Further, $\overline{QI}$ a leg of both ΔQIF and ΔQID.
- $\Delta QIF \cong \Delta QID$ by the Hypotenuse-Leg Theorem for the Congruence of Right Triangles.
- $\angle IQF \cong \angle IQD$ because corresponding parts of congruent triangles are congruent.
- $\angle IQD \cong \angle I'QD'$ because they are vertical angles.

Finally, we note that $\angle IQF \cong \angle I'QD'$ by the transitive property of congruence. *QED*