Math Handbook

of Formulas, Processes and Tricks

(www.mathguy.us)

High School Equivalency (GED) Exam

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Pre-Algebra

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Pre-Algebra
Divisibility Rules

The following rules can be used to determine whether a number is divisible by other numbers. This is particularly useful in reducing fractions to lowest terms because the rules can be used to test whether both the numerator and denominator are divisible by the same number.

n	**A number is divisible by "n" if and only if:**	**Examples**
2	It is even, i.e., if it ends in 0, 2, 4, 6 or 8.	16 (even because it end in a 6) 948 (even because it ends in an 8)
3	The sum of its digits is divisible by 3. You may apply this test multiple times if necessary.	42 (4+2=6) 948 (9+4+8=21, then 2+1=3)
4	The number formed by its last 2 digits is divisible by 4.	332 (32÷4=8) 1,908 (08÷4=2)
5	It ends in a 0 or 5.	905 (ends in a 5) 384,140 (ends in a 0)
6	It is divisible by both 2 and 3.	36 (it is even and 3+6=9) 948 (it is even and 9+4+8=21)
7	Double the last digit and subtract it from the rest of the number. If the result is divisible by 7, so is the original number. You may apply this test multiple times if necessary.	868 (86-[2·8]=70, and 70÷7=10) 2,345 (234-[2·5]=224, then apply again: 22-[2·4]=14, and 14÷7=2)
8	The number formed by its last 3 digits is divisible by 8.	92,104 (104÷8=13) 727,520 (520÷8=65)
9	The sum of its digits is divisible by 9. You may apply this test multiple times if necessary.	2,385 (2+3+8+5=18, then 1+8=9) 89,487 (8+9+4+8+7=36, then 3+6=9)
10	It ends in a 0.	370 (ends in a 0) 345,890 (ends in a 0)
11	The alternating sum and difference of its digits is divisible by 11.	374 (3-7+4=0) 9,482 (9-4+8-2=11)
12	It is divisible by both 3 and 4.	996 (9+9+6=24 and 96÷4=24) 1,344 (1+3+4+4=12 and 44÷4=11)

Note: **0** is divisible by every number except itself.

Pre-Algebra
Prime Numbers

Definitions

A prime number is a natural number (i.e., a positive integer) that has no factors other than 1 and itself. The prime numbers less than 50 are:

$$\mathbf{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}$$

A composite number is a natural number that is not prime.

Prime Factorization

Every natural number has a unique prime factorization. This means that if you factor the number until all you have left are prime numbers, there is only one representation of the number in this form (ignoring the order of the factors). By mathematical convention, the prime factorization of a number is expressed as a product of its prime factors in numerical order, from low to high, with exponents on factors that are repeated.

Examples: $40 = 2^3 \cdot 5$ $\quad$ $330 = 2 \cdot 3 \cdot 5 \cdot 11$ $\quad$ $637 = 7^2 \cdot 13$

Deriving a Prime Factorization

To derive the unique prime factorization of a number n:

- Divide the number by 2 as many times as 2 will go into the number.
- Move up to the next prime number and repeat the process.
- Repeat the previous step until all of the factors are prime.

Note: In a prime factorization, all of the factors will be less than $\sqrt{n}$.

Examples: Find the prime factorizations of 336, 1000, and 2160.

$$\begin{aligned} 336 &= 2 \cdot 168 \\ &= 2 \cdot 2 \cdot 84 \\ &= 2 \cdot 2 \cdot 2 \cdot 42 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 21 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 \\ &= 2^4 \cdot 3 \cdot 7 \end{aligned}$$

$$\begin{aligned} 1000 &= 2 \cdot 500 \\ &= 2 \cdot 2 \cdot 250 \\ &= 2 \cdot 2 \cdot 2 \cdot 125 \\ &= 2 \cdot 2 \cdot 2 \cdot 5 \cdot 25 \\ &= 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 \\ &= 2^3 \cdot 5^3 \end{aligned}$$

$$\begin{aligned} 2160 &= 2 \cdot 1080 \\ &= 2 \cdot 2 \cdot 540 \\ &= 2 \cdot 2 \cdot 2 \cdot 270 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 135 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 45 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 15 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \\ &= 2^4 \cdot 3^3 \cdot 5 \end{aligned}$$

Pre-Algebra
Prime Factor Trees

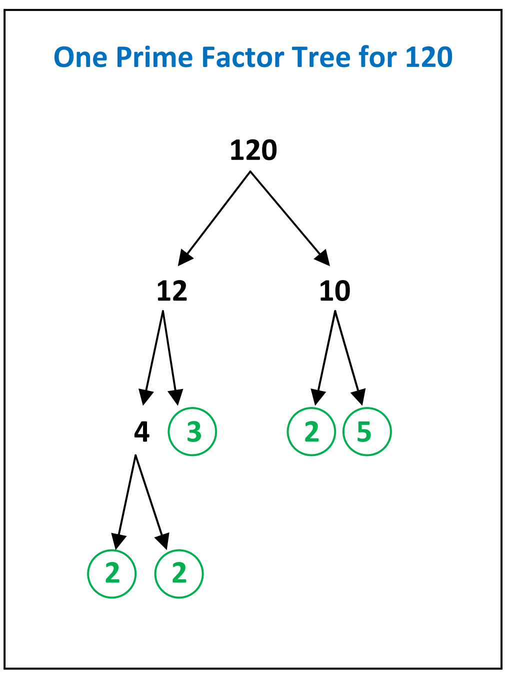
What is a Prime Factor Tree?

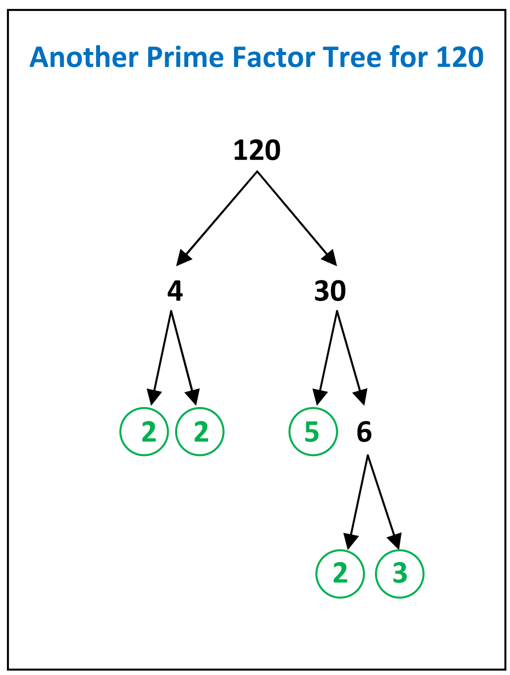
A **Prime Factor Tree** is a device that can be used to find the prime factors of a number. Even though each number has a unique set of prime factors, most numbers do not have a unique prime factor tree. The nice thing about a tree is that you can work with any factors of the number, and by the time you have finished, you have found its unique set of prime factors.

To develop a prime factor tree:

- Write the number to be factored at the top of the tree.
- Beneath the number, write a pair of factors that multiply to get the number.
- Repeat the above step until all of the factors are prime.
- It is useful to identify the prime factors you develop in some manner, like circling them.
- Collect all of the prime factors to obtain the prime factorization of the number.

Examples:





In both cases, the prime factorization of 120 is determined to be: $\mathbf{120 = 2^3 \cdot 3 \cdot 5}$

Notice that the two trees in the examples obtain the same result even though they take different paths to get that result. Other paths are possible as well. The important thing is the result, not the path.

Pre-Algebra
GCD and LCM

Simple methods for finding the Greatest Common Divisor (GCD) and the Least Common Multiple (LCM) are related, as shown below. Both involve developing a table of prime factors for the numbers in question. The methods are best illustrated by example.

Greatest Common Divisor (GCD)

Example A: Find the GCD of 180 and 105.

Step 1: Calculate the prime factors of each number and enter them into a small table:

180 =	2	x 2	x 3	x 3	x 5	
105 =			3		x 5	x 7
GCD =			3		x 5	

So, GCD = 3 x 5 = 15.

Step 2: Line up the prime factors so that those common to all of the numbers are in the same column.

Step 3: Bring any factors that show up for every number (i.e., that fill the column) below the line.

Step 4: Multiply all of the numbers below the line to obtain the GCD.

Example B: Find the GCD of 140, 210 and 462.

140 =	2	x 2		x 5	x 7	
210 =	2		x 3	x 5	x 7	
462 =	2		x 3		x 7	x 11
GCD =	2				x 7	

So, GCD = 2 x 7 = 14.

Example C: Find the GCD of 32 and 27.

32 =	2	x 2	x 2	x 2	x 2			
27 =						3	x 3	x 3

GCD = (there are no common factors)

So, GCD = 1.

Calculating a Number's Prime Factors

In order to calculate the prime factors of a number, simply begin dividing it by primes, starting with 2 and working higher until all factors are primes.

Examples: Find the prime factors of …

462	180
462 = 2 x 231	180 = 2 x 90
= 2 x 3 x 77	180 = 2 x 2 x 45
= 2 x 3 x 7 x 11	180 = 2 x 2 x 3 x 15
	180 = 2 x 2 x 3 x 3 x 5

If no common prime factors exist, GCD = 1 and the numbers are said to be ***relatively prime***. Since 27 and 32 have no common prime factors, they are ***relatively prime***.

Pre-Algebra
GCD and LCM

Least Common Multiple (LCM)

Example A: Find the LCM of 12 and 18.

Step 1: Calculate the prime factors of each number and enter them into a small table:

12 =	2	x 2	x 3	
18 =	2		x 3	x 3
LCM =	2	x 2	x 3	x 3

So the LCM = 2 x 2 x 3 x 3 = 36

Step 2: Line up the prime factors so that those common to all of the numbers are in the same column.

Step 3: Bring one factor from every column below the line.

Step 4: Multiply all of the numbers below the line to obtain the LCM.

Example B: Find the LCM of 6, 8 and 18.

6 =	2			x 3	
8 =	2	x 2	x 2		
18 =	2			x 3	x 3
LCM =	2	x 2	x 2	x 3	x 3

So, the LCM = 2 x 2 x 2 x 3 x 3 = 72.

Lowest Common Denominator (LCD)

When fractions with different denominators are to be added or subtracted, it is necessary to find the Lowest Common Denominator. The LCD is essentially the Least Common Multiple of the denominators in question. Consider this problem:

Example: **Calculate:** $\frac{5}{12}-\frac{7}{18}$. In Example A, the LCM of 12 and 18 was calculated to be 36. To determine which fractional name for 1 must be multiplied by each fraction to obtain a common denominator, we look for the missing numbers in each row. From Example A above:

12 =	2	x 2	x 3	□ ← missing a "3"
18 =	2	□ ← missing a "2"	x 3	x 3
LCM =	2	x 2	x 3	x 3 = 36

The missing number in the row for 12 is 3. Therefore, we use $\frac{3}{3}$ as a multiplier for $\frac{5}{12}$. Similarly, we use $\frac{2}{2}$ as a multiplier for $\frac{7}{18}$.

Result: $\left(\frac{3}{3}\right)\left(\frac{5}{12}\right)-\left(\frac{7}{18}\right)\left(\frac{2}{2}\right) = \frac{15}{36}-\frac{14}{36} = \frac{1}{36}$

Pre-Algebra
Metric Measures

King Henry

The following mnemonic device can be used to remember the order of metric measurements:

	King	Henry	Died	By	Drinking	Chocolate	Milk
Length	km	hm	dkm	meters	dm	cm	mm
Mass	kg	hg	dkg	grams	dg	cg	mg
Volume	kl	hl	dkl	liters	dl	cl	ml

← In moving from right to left:

For each position traversed,

- Divide by 10, or
- Move the decimal one place to the left.
- Add zeroes if needed.

→ In moving from left to right:

For each position traversed,

- Multiply by 10, or
- Move the decimal one place to the right.
- Add zeroes if needed.

In the mnemonic, the "b" in "by" stands for "base unit"; this is the unit that all others are based upon. The base units above are meters, grams, and liters.

Note: 1 ml = 1 cubic centimeter

The prefixes to the base unit, along with their meanings are:

$k = kilo$	$1{,}000$
$h = hecto$	100
$dk = deka$	10

$d = deci$	$\frac{1}{10}$
$c = centi$	$\frac{1}{100}$
$m = milli$	$\frac{1}{1{,}000}$

Examples:

$1\ km = 100{,}000\ cm$	Add 5 zeroes to the right (for 5 positions moved in the above chart).
$32\ mg = .000\ 032\ kilograms$	Move the decimal 6 places to the left (for the 6 positions moved to the left in the above chart).
$2.5\ liters = 2{,}500\ ml$	Move the decimal 3 places to the right (for the 3 positions moved to the right in the above chart).

Pre-Algebra
Measures and Weights – U.S. Conversions

Distance
1 foot = 12 inches
1 yard = 3 feet = 36 inches
1 fathom = 2 yards = 6 feet
1 rod = 5.5 yards = 16.5 feet
1 furlong = 40 rods = 220 yards
1 mile = 8 furlongs = 1,760 yards = 5,280 feet
1 league = 3 miles = 24 furlongs
1 acre = 43,560 square feet
1 square mile = 640 acres

Weight
1 pennyweight = 24 grains
1 dram = 27.344 grains
1 ounce = 16 drams = 437.5 grains
1 Troy ounce = 20 pennyweight = 480 grains
1 pound = 16 ounces = 7,000 grains
1 stone = 14 pounds
1 ton = 2,000 pounds
1 long ton = 2,240 pounds

Time
1 minute = 60 seconds
1 hour = 60 minutes = 3,600 seconds
1 day = 24 hours = 1,440 minutes
1 week = 7 days = 168 hours
1 fortnight = 2 weeks = 14 days
1 month = 4 1/3 weeks
1 year = 12 months = 52 weeks
1 year = 365 1/4 days

Capacity
1 fluid dram = 60 minims
1 fluid ounce = 8 fluid drams
1 gill = 4 fluid ounces
1 cup = 2 gills = 8 fluid ounces
1 pint = 2 cups = 16 fluid ounces
1 quart = 2 pints = 4 cups
1 gallon = 4 quarts = 16 cups
1 peck = 2 gallons
1 bushel = 4 pecks = 8 gallons

Pre-Algebra
Decimal Conversions

1/2	.50	50%
1/3	.333	33.3%
2/3	.667	66.7%
1/4	.25	25%
2/4	.50	50%
3/4	.75	75%
1/5	.20	20%
2/5	.40	40%
3/5	.60	60%
4/5	.80	80%
1/6	.167	16.7%
2/6	.333	33.3%
3/6	.500	50.0%
4/6	.667	66.7%
5/6	.833	83.3%
1/7	.142857	14.2857%
2/7	.285714	28.5714%
3/7	.428571	42.8571%
4/7	.571429	57.1429%
5/7	.714286	71.4286%
6/7	.857143	85.7143%

1/8	.125	12.5%
2/8	.250	25.0%
3/8	.375	37.5%
4/8	.500	50.0%
5/8	.625	62.5%
6/8	.750	75.0%
7/8	.875	87.5%
1/9	.111	11.1%
2/9	.222	22.2%
3/9	.333	33.3%
4/9	.444	44.4%
5/9	.556	55.6%
6/9	.667	66.7%
7/9	.778	77.8%
8/9	.889	88.9%
1/10	.1	10%
2/10	.2	20%
3/10	.3	30%
4/10	.4	40%
5/10	.5	50%
6/10	.6	60%
7/10	.7	70%
8/10	.8	80%
9/10	.9	90%

1/11	.09091	9.091%
1/12	.08333	8.333%
1/13	.07692	7.692%
1/14	.07143	7.143%
1/15	.06667	6.667%
1/16	.06250	6.250%
1/17	.05882	5.882%
1/18	.05556	5.556%
1/19	.05263	5.263%
1/20	.05000	5.000%

Pre-Algebra
Applying a Percent Increase

It is common in mathematics to work with percent increases. An example of an everyday application of this is the sales tax you pay in the local store. Sales tax is expressed in the form of a percent increase.

Applying a Percent Increase

There are two methods for working with percent increases. Use the one you like best.

Method 1:

- Start with the amount before increase (i.e., the original amount).
- Calculate the amount of the increase.
- Add the original amount and the amount of the increase to obtain the final amount.

$$\binom{increase}{amount} = \binom{original}{amount} \cdot \binom{percent}{increase}$$

$$\binom{final}{amount} = \binom{original}{amount} + \binom{increase}{amount}$$

An advantage of this approach is that you calculate the amount of the increase. Sometimes, this is an important value to know.

Example: What do you get when you increase 150 by 10%?

$$Increase\ Amount = 10\% \cdot 150 = 15$$

$$Final\ Amount = 150 + 15 = 165$$

Method 2:

- Add the percent increase to 100%.
- Multiply the original amount by this new percentage to obtain the final amount.

$$\binom{total}{percent} = 100\% + \binom{percent}{increase}$$

$$\binom{final}{amount} = \binom{original}{amount} \cdot \binom{total}{percent}$$

This approach may be easier and has extensive business applications.

Example: What do you get when you increase 150 by 10%?

$$Total\ Percent = 100\% + 10\% = 110\% = 1.1$$

$$Final\ Amount = 150 \cdot 1.1 = 165$$

Pre-Algebra
Applying a Percent Decrease

It is common in mathematics to work with percent decreases. In a store you may see a sign that says "Sale – 40% off." In such a case, you may want to calculate the sale price.

Applying a Percent Decrease

There are two methods for working with percent decreases. Use the one you like best.

Method 1:

- Start with the amount before decrease (i.e., the original amount).
- Calculate the amount of the decrease.
- Subtract the amount of the decrease from the original amount to obtain the final amount.

$$\begin{pmatrix} decrease \\ amount \end{pmatrix} = \begin{pmatrix} original \\ amount \end{pmatrix} \cdot \begin{pmatrix} percent \\ decrease \end{pmatrix}$$

$$\begin{pmatrix} final \\ amount \end{pmatrix} = \begin{pmatrix} original \\ amount \end{pmatrix} - \begin{pmatrix} decrease \\ amount \end{pmatrix}$$

An advantage of this approach is that you calculate the amount of the decrease. Sometimes, this is an important value to know (e.g., how much money did you save?).

Example: What do you get when you decrease 150 by 40%?

$$Decrease\ Amount = 40\% \cdot 150 = 60$$

$$Final\ Amount = 150 - 60 = 90$$

Method 2:

- Subtract the percent increase from 100%.
- Multiply the original amount by this new percentage to obtain the final amount.

$$\begin{pmatrix} total \\ percent \end{pmatrix} = 100\% - \begin{pmatrix} percent \\ decrease \end{pmatrix}$$

$$\begin{pmatrix} final \\ amount \end{pmatrix} = \begin{pmatrix} original \\ amount \end{pmatrix} \cdot \begin{pmatrix} total \\ percent \end{pmatrix}$$

This approach may be easier and has the same form as the formula for percent increase. It also has extensive business applications.

Example: What do you get when you decrease 150 by 40%?

$$Total\ Percent = 100\% - 40\% = 60\% = 0.6$$

$$Final\ Amount = 150 \cdot 0.6 = 90$$

Pre-Algebra
Calculating Percent Increases and Decreases

Many times, you have the original amount and the final amount after either an increase or decrease in value. You may want to calculate the percent of that increase or decrease.

Percent Increase

Given a starting amount and a final amount,

$$\binom{increase}{amount} = \binom{final}{amount} - \binom{original}{amount}$$

$$\binom{percent}{increase} = \binom{amount\ of}{increase} \div \binom{original}{amount}$$

Example: A stock increases in value from \$80 to \$96; what percent has it increased?

$$increase\ amount = \$96 - \$80 = \$16$$

$$percent\ increase = \$16 \div \$80 = .20 = 20\%$$

Percent Decrease

Given a starting amount and a final amount,

$$\binom{decrease}{amount} = \binom{original}{amount} - \binom{final}{amount}$$

$$\binom{percent}{decrease} = \binom{amount\ of}{decrease} \div \binom{original}{amount}$$

Example: A stock decreases in value from \$80 to \$68; what percent has it increased?

$$decrease\ amount = \$80 - \$68 = \$12$$

$$percent\ decrease = \$12 \div \$80 = .15 = 15\%$$

Notice the following:

- You calculate both an increase and a decrease as the difference between the original and final amounts.
- The percent change is always calculated as the *amount of the change* divided by the *original amount*.

Pre-Algebra
Estimating Square Roots

Square Roots of Perfect Squares

The square root of a perfect square can be read right off the table to the right. For example,

$$\sqrt{36} = 6 \quad \textit{because} \quad 6^2 = 36$$

It is worthwhile to memorize the perfect squares in this table. They occur very frequently in math from 7th grade and up.

Square Roots of Other Numbers

Square roots of numbers other than perfect squares can be estimated with a process called interpolation. **To calculate $\sqrt{n}$**:

- Find where n fits between perfect squares in the right hand column of the table to the right.
- Determine the corresponding square roots of the perfect squares above and below n.
- Interpolate between the two square roots in the previous step based on where n lies between the perfect squares.

Example:

Estimate $\sqrt{127}$

127 lies between 121 and 144 in the table to the right.

Line up the three square roots and perfect squares in a table:

		Square Root	Perfect Square		
1		**11**	**121**		23
	$x-11$	$x = \sqrt{127}$	**127**	6	
		12	**144**		

Then, solve the proportion: $\frac{x-11}{1} = \frac{6}{23}$

Cross multiply: $23x - 253 = 6 \quad \Longrightarrow \quad 23x = 259$

Finally, $x = \frac{259}{23} = 11.26 \quad \Longrightarrow \quad \sqrt{127} = \mathbf{11.26}$ by interpolation

Using a calculator, the actual value, to 2 decimals, is: $\sqrt{127} = 11.27$ by calculator

Table of Squares

$1^2 = 1$

$2^2 = 4$

$3^2 = 9$

$4^2 = 16$

$5^2 = 25$

$6^2 = 36$

$7^2 = 49$

$8^2 = 64$

$9^2 = 81$

$10^2 = 100$

$11^2 = 121$

$12^2 = 144$

$13^2 = 169$

$14^2 = 196$

$15^2 = 225$

$16^2 = 256$

$17^2 = 289$

$18^2 = 324$

$19^2 = 361$

$20^2 = 400$

Pre-Algebra
Powers of 10

Uses of Powers of 10

Powers of 10 are useful in mathematics and science. In particular, they are used in scientific notation to express very large numbers and very small numbers without using up all the space a bunch of zeroes would take. Numbers with a lot of zeroes are also hard to grasp, whereas powers of 10 are relatively easy to grasp.

Negative Powers of 10

For negative powers of 10, the number of zeroes before the 1, including one zero to the left of the decimal point, is equal to the exponent (disregarding the negative sign).

Zero Power of 10

$10^0 = 1$ *(notice, no zeroes to the left or right of the 1)*

Positive Powers of 10

For positive powers of 10, the number of zeroes after the 1 is equal to the exponent.

Fun Only – Special Cases

There are two special cases for powers of 10 that mathematicians have defined. For very big numbers, mathematicians have defined the googol and the googolplex. These are not to be confused with Google, the internet search engine; they are spelled differently.

They are defined as:

$googol = 10^{100}$ (a 1 followed by 100 zeroes)

$googolplex = 10^{\text{googol}}$ (a 1 followed by googol zeroes)

Maybe you can create your own name for:

$10^{\text{googolplex}}$ (a 1 followed by googolplex zeroes)

Powers of 10

$10^{-6} = 0.000\ 001$

$10^{-5} = 0.000\ 01$

$10^{-4} = 0.000\ 1$

$10^{-3} = 0.001$

$10^{-2} = 0.01$

$10^{-1} = 0.1$

$10^0 = 1$

$10^1 = 10$

$10^2 = 100$

$10^3 = 1{,}000$

$10^4 = 10{,}000$

$10^5 = 100{,}000$

$10^6 = 1{,}000{,}000$

$10^{100} = \text{googol}$

$10^{\text{googol}} = googolplex$

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Algebra

Linear Patterns

Recognizing Linear Patterns

The first step to recognizing a pattern is to arrange a set of numbers in a table. The table can be either horizontal or vertical. Here, we consider the pattern in a horizontal format. More advanced analysis generally uses the vertical format.

Consider this pattern:

x-value	0	1	2	3	4	5
y-value	6	9	12	15	18	21

To analyze the pattern, we calculate differences of successive values in the table. These are called first differences. If the first differences are constant, we can proceed to converting the pattern into an equation. If not, we do not have a linear pattern. In this case, we may choose to continue by calculating differences of the first differences, which are called second differences, and so on until we get a pattern we can work with.

In the example above, we get a constant set of first differences, which tells us that the pattern is indeed linear.

x-value	0	1	2	3	4	5
y-value	6	9	12	15	18	21

First Differences	3	3	3	3	3

Converting a Linear Pattern to an Equation

Creating an equation from the pattern is easy if you have constant differences and a y-value for x = 0. In this case,

- The equation takes the form $\boldsymbol{y = mx + b}$, where
- ***"m"*** is the constant difference from the table, and
- ***"b"*** is the y-value when x = 0.

In the example above, this gives us the equation: $\boldsymbol{y = 3x + 6}$.

> **Note:** If the table does not have a value for x=0, you can still obtain the value of **"b"**. Simply extend the table left or right until you have an x-value of 0; then use the first differences to calculate what the corresponding y-value would be. This becomes your value of **"b"**.

Finally, it is a good idea to test your equation. For example, if $x = 4$, the above equation gives $y = (3 \cdot 4) + 6 = 18$, which is the value in the table. So we can be pretty sure our equation is correct.

Algebra
Identifying Number Patterns

When looking at patterns in numbers, is is often useful to take differences of the numbers you are provided. If the first differences are not constant, take differences again.

n	Δ
-3	
	2
-1	
	2
1	
	2
3	
	2
5	
	2
7	

When first differences are constant, the pattern represents a linear equation. In this case, the equation is: $y = 2x - 5$. The constant difference is the coefficient of x in the equation.

n	Δ	$Δ^2$
2		
	3	
5		2
	5	
10		2
	7	
17		2
	9	
26		2
	11	
37		

When second differences are constant, the pattern represents a quadratic equation. In this case, the equation is: $y = x^2 + 1$. The constant difference, divided by 2, gives the coefficient of x^2 in the equation.

When taking successive differences yields patterns that do not seem to level out, the pattern may be either exponential or recursive.

n	Δ	$Δ^2$
5		
	2	
7		2
	4	
11		4
	8	
19		8
	16	
35		16
	32	
67		

In the pattern to the left, notice that the first and second differences are the same. You might also notice that these differences are successive powers of 2. This is typical for an exponential pattern. In this case, the equation is: $y = 2^x + 3$.

n	Δ	$Δ^2$
2		
	1	
3		1
	2	
5		1
	3	
8		2
	5	
13		3
	8	
21		

In the pattern to the left, notice that the first and second differences appear to be repeating the original sequence. When this happens, the sequence may be recursive. This means that each new term is based on the terms before it. In this case, the equation is: $y_n = y_{n-1} + y_{n-2}$, meaning that to get each new term, you add the two terms before it.

Algebra
Completing Number Patterns

The first step in completing a number pattern is to identify it. Then, work from the right to the left, filling in the highest order differences first and working backwards (left) to complete the table. Below are two examples.

Consider in the examples the sequences of six numbers which are provided to the student. You are asked to find the ninth term of each sequence.

Example 1

n
-1
6
25
62
123
214

Example 2

n
2
3
5
8
13
21

Step 1: Create a table of differences. Take successive differences until you get a column of constant differences (Example 1) or a column that appears to repeat a previous column of differences (Example 2).

n	Δ	$Δ^2$	$Δ^3$
-1			
	7		
6		12	
	19		6
25		18	
	37		6
62		24	
	61		6
123		30	
	91		
214			

n	Δ	$Δ^2$	$Δ^3$
2			
	1		
3		1	
	2		0
5		1	
	3		1
8		2	
	5		1
13		3	
	8		
21			

Step 2: In the last column of differences you created, continue the constant differences (Example 1) or the repeated differences (Example 2) down the table. Create as many entries as you will need to solve the problem. For example, if you are given 6 terms and asked to find the 9th term, you will need 3 (= 9 - 6) additional entries in the last column.

n	Δ	$Δ^2$	$Δ^3$
-1			
	7		
6		12	
	19		6
25		18	
	37		6
62		24	
	61		6
123		30	
	91		**6**
214			
			6
			6

n	Δ	$Δ^2$	$Δ^3$
2			
	1		
3		1	
	2		0
5		1	
	3		1
8		2	
	5		1
13		3	
	8		**2**
21			
			3
			5

Step 3: Work backwards (from right to left), filling in each column by adding the differences in the column to the right.

In the example to the left, the calculations are performed in the following order:

Column $Δ^2$: 30 + 6 = 36; 36 + 6 = 42; 42 + 6 = 48

Column Δ: 91 + 36 = 127; 127 + 42 = 169; 169 + 48 = 217

Column n: 214 + 127 = 341; 341 + 169 = 510; 510 + 217 = **727**

n	Δ	$Δ^2$	$Δ^3$
-1			
	7		
6		12	
	19		6
25		18	
	37		6
62		24	
	61		6
123		30	
	91		**6**
214		**36**	
	127		**6**
341		**42**	
	169		**6**
510		**48**	
	217		
727			

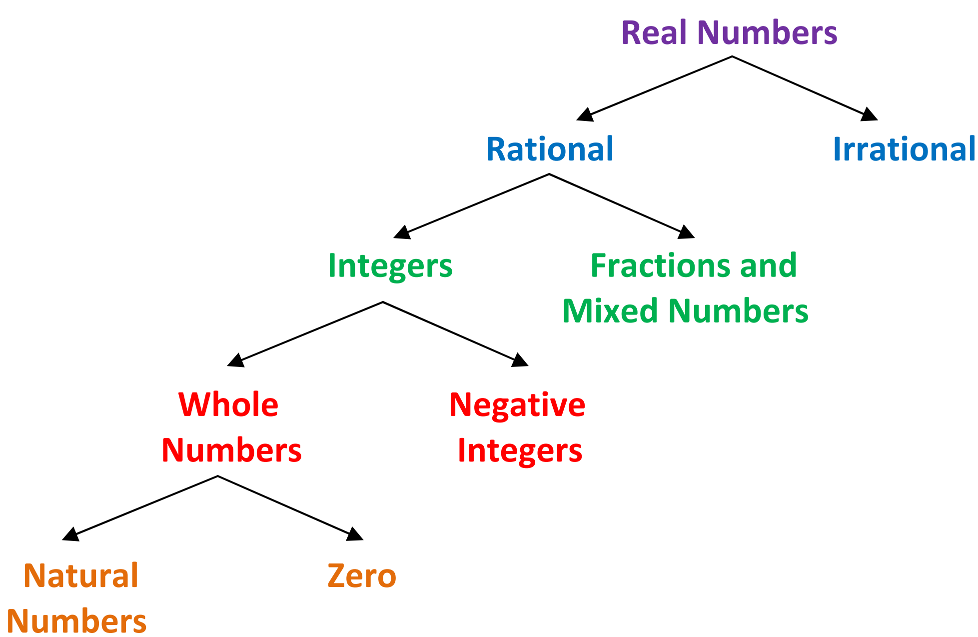
n	Δ	$Δ^2$	$Δ^3$
2			
	1		
3		1	
	2		0
5		1	
	3		1
8		2	
	5		1
13		3	
	8		**2**
21		**5**	
	13		**3**
34		**8**	
	21		**5**
55		**13**	
	34		
89			

The final answers to the examples are the ninth items in each sequence, the items in **bold red.**

Algebra
Basic Number Sets

Number Set	Definition	Examples
Natural Numbers (or, Counting Numbers)	Numbers that you would normally count with.	1, 2, 3, 4, 5, 6, …
Whole Numbers	Add the number *zero* to the set of Natural Numbers	0, 1, 2, 3, 4, 5, 6, …
Integers	Whole numbers plus the set of negative Natural Numbers	… -3, -2, -1, 0, 1, 2, 3, …
Rational Numbers	Any number that can be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.	All integers, plus fractions and mixed numbers, such as: $-\frac{2}{3}$, $\frac{17}{6}$, $3\frac{4}{5}$
Real Numbers	Any number that can be written in decimal form, even if that form is infinite.	All rational numbers plus roots and some others, such as: $\sqrt{2}$, $\sqrt[3]{12}$, π, e

Basic Number Set Tree



Algebra
Operating with Real Numbers

Absolute Value

The absolute value of something is the distance it is from zero. The easiest way to get the absolute value of a number is to eliminate its sign. Absolute values are always positive or 0.

$$|-5| = 5 \qquad |3| = 3 \qquad |0| = 0 \qquad \left|-\frac{3}{4}\right| = \frac{3}{4} \qquad |1.5| = 1.5$$

Adding and Subtracting Real Numbers

Adding Numbers with the Same Sign:

- Add the numbers without regard to sign.
- Give the answer the same sign as the original numbers.
- Examples:

$$(-6) + (-3) = -9$$
$$12 + 6 = 18$$

Adding Numbers with Different Signs:

- Ignore the signs and subtract the smaller number from the larger one.
- Give the answer the sign of the number with the greater absolute value.
- Examples:

$$(-6) + 3 = -3$$
$$(-7) + 11 = 4$$

Subtracting Numbers:

- Change the sign of the number or numbers being subtracted.
- Add the resulting numbers.
- Examples:

$$(-6) - (-3) = (-6) + 3 = -3$$
$$13 - 4 = 13 + (-4) = 9$$

Multiplying and Dividing Real Numbers

Numbers with the Same Sign:

- Multiply or divide the numbers without regard to sign.
- Give the answer a "+" sign.
- Examples:

$$(-6) \cdot (-3) = +18 = 18$$
$$12 \div 3 = +4 = 4$$

Numbers with Different Signs:

- Multiply or divide the numbers without regard to sign.
- Give the answer a "-" sign.
- Examples:

$$(-6) \cdot (3) = -18$$
$$12 \div (-3) = -4$$

Algebra
Properties of Algebra

Properties of Addition and Multiplication. For any real numbers **a**, **b**, and **c**:

Property	Definition for Addition	Definition for Multiplication
Closure Property	$a + b$ is a real number	$a \cdot b$ is a real number
Identity Property	$a + 0 = 0 + a = a$	$a \cdot 1 = 1 \cdot a = a$
Inverse Property	$a + (-a) = (-a) + a = 0$	$For\ a \neq 0,\ \ a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$
Commutative Property	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative Property	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Distributive Property	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	

Properties of Zero. For any real number **a**:

Multiplication by 0	$a \cdot 0 = 0 \cdot a = 0$
0 Divided by Something	$For\ a \neq 0,\ \frac{0}{a} = 0$
Division by 0	$\frac{a}{0}$ is undefined (even if $a = 0$)

Algebra
Probability and Odds

Probability

Probability is a measure of the likelihood that an event will occur. It depends on the number of outcomes that represent the event and the total number of possible outcomes. In equation terms,

$$P(event) = \frac{number\ of\ outcomes\ representing\ the\ event}{number\ of\ total\ possible\ outcomes}$$

Example 1: The probability of a flipped coin landing as a head is 1/2. There are two equally likely events when a coin is flipped – it will show a head or it will show a tail. So, there is one chance out of two that the coin will show a head when it lands.

$$P(head) = \frac{1\ outcome\ of\ a\ head}{2\ total\ possible\ outcomes} = \frac{1}{2}$$

Example 2: In a jar, there are 15 blue marbles, 10 red marbles and 7 green marbles. What is the probability of selecting a red marble from the jar? In this example, there are 32 total marbles, 10 of which are red, so there is a 10/32 (or, when reduced, 5/16) probability of selecting a red marble.

$$P(red\ marble) = \frac{10\ red\ marbles}{32\ total\ marbles} = \frac{10}{32} = \frac{5}{16}$$

Odds

Odds are similar to probability, except that we measure the number of chances that an event will occur relative to the number of chances that the event will not occur.

$$Odds(event) = \frac{number\ of\ outcomes\ representing\ the\ event}{number\ of\ outcomes\ NOT\ representing\ the\ event}$$

In the above examples,

$$Odds(head) = \frac{1\ outcome\ of\ a\ head}{1\ outcome\ of\ a\ tail} = \frac{1}{1} \qquad Odds(red\ marble) = \frac{10\ red\ marbles}{22\ other\ marbles} = \frac{10}{22} = \frac{5}{11}$$

- Note that the numerator and the denominator in an odds calculation add to the total number of possible outcomes in the denominator of the corresponding probability calculation.

- To the beginning student, the concept of odds is not as intuitive as the concept of probabilities; however, they are used extensively in some environments.

Algebra
Probability with Dice

Single Die

Probability with a single die is based on the number of chances of an event out of 6 possible outcomes on the die. For example:

$$P(2) = \frac{1}{6} \qquad P(odd\ number) = \frac{3}{6} = \frac{1}{2} \qquad P(number < 5) = \frac{4}{6} = \frac{2}{3}$$

Two Dice

Probability with two dice is based on the number of chances of an event out of 36 possible outcomes on the dice. The following table of results when rolling 2 dice is helpful in this regard:

	1st Die					
2nd Die	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The probability of rolling a number with two dice is the number of times that number occurs in the table, divided by 36. Here are the probabilities for all numbers 2 to 12.

$$P(2) = \frac{1}{36} \qquad P(5) = \frac{4}{36} = \frac{1}{9} \qquad P(8) = \frac{5}{36} \qquad P(11) = \frac{2}{36} = \frac{1}{18}$$

$$P(3) = \frac{2}{36} = \frac{1}{18} \qquad P(6) = \frac{5}{36} \qquad P(9) = \frac{4}{36} = \frac{1}{9} \qquad P(12) = \frac{1}{36}$$

$$P(4) = \frac{3}{36} = \frac{1}{12} \qquad P(7) = \frac{6}{36} = \frac{1}{6} \qquad P(10) = \frac{3}{36} = \frac{1}{12}$$

$$P(odd\ number) = \frac{18}{36} = \frac{1}{2} \qquad P(number\ divisible\ by\ 3) = \frac{2+5+4+1}{36} = \frac{12}{36} = \frac{1}{3}$$

$$P(even\ number) = \frac{18}{36} = \frac{1}{2} \qquad P(number\ divisible\ by\ 4) = \frac{3+5+1}{36} = \frac{9}{36} = \frac{1}{4}$$

$$P(number\ divisible\ by\ 6) = \frac{5+1}{36} = \frac{6}{36} = \frac{1}{6}$$

Algebra
Combinations

Single Category Combinations

The number of combinations of items selected from a set, several at a time, can be calculated relatively easily using the following technique:

> ***Technique:*** Create a ratio of two products. In the numerator, start with the number of total items in the set, and count down so the total number of items being multiplied is equal to the number of items being selected. In the denominator, start with the number of items being selected and count down to 1.

Example: How many combinations of 3 items can be selected from a set of 8 items? Answer: $\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$	**Example:** How many combinations of 4 items can be selected from a set of 13 items? Answer: $\frac{13 \cdot 12 \cdot 11 \cdot 10}{4 \cdot 3 \cdot 2 \cdot 1} = 715$	**Example:** How many combinations of 2 items can be selected from a set of 30 items? Answer: $\frac{30 \cdot 29}{2 \cdot 1} = 435$

Multiple Category Combinations

When calculating the number of combinations that can be created by selecting items from several categories, the technique is simpler:

> ***Technique:*** Multiply the numbers of items in each category to get the total number of possible combinations.

Example: How many different pizzas could be created if you have 3 kinds of dough, 4 kinds of cheese and 8 kinds of toppings? Answer: $3 \cdot 4 \cdot 8 = 96$	**Example:** How many different outfits can be created if you have 5 pairs of pants, 8 shirts and 4 jackets? Answer: $5 \cdot 8 \cdot 4 = 160$	**Example:** How many designs for a car can be created if you can choose from 12 exterior colors, 3 interior colors, 2 interior fabrics and 5 types of wheels? Answer: $12 \cdot 3 \cdot 2 \cdot 5 = 360$

Algebra
Statistical Measures

Statistical measures help describe a set of data. A definition of a number of these is provided in the table below:

Concept	Description	Calculation	Example 1	Example 2
Data Set	Numbers		35, 35, 37, 38, 45	15, 20, 20, 22, 25, 54
Mean	Average	Add the values and divide the total by the number of values	$\frac{35+35+37+38+45}{5}=38$	$\frac{15+18+22+22+25+54}{6}=26$
Median[1]	Middle	Arrange the values from low to high and take the middle value[1]	37	21[1]
Mode	Most	The value that appears most often in the data set	35	20
Range	Size	The difference between the highest and lowest values in the data set	45 – 35 = 10	54 – 15 = 39
Outliers[2]	Oddballs	Values that look very different from the other values in the data set	none	54

Notes:

(1) If there are an even number of values, the median is the average of the two middle values. In Example 2, the median is 21, which is the average of 20 and 22.

(2) The question of what constitutes an outlier is not always clear. Although statisticians seek to minimize subjectivity in the definition of outliers, different analysts may choose different criteria for the same data set.

Algebra
Introduction to Functions

Definitions

- A **Relation** is a relationship between variables, usually expressed as an equation.
- In a typical ***x-y*** equation, the **Domain** of a relation is the set of ***x***-values for which ***y***-values can be calculated. For example, in the relation $y = \sqrt{x}$ the domain is $x \geq 0$ because these are the values of ***x*** for which a square root can be taken.
- In a typical ***x-y*** equation, the **Range** of a relation is the set of ***y***-values that result for all values of the domain. For example, in the relation $y = \sqrt{x}$ the range is $y \geq 0$ because these are the values of ***y*** that result from all the values of ***x***.
- A **Function** is a relation in which each element in the domain has only one corresponding element in the range.
- A **One-to-One Function** is a function in which each element in the range is produced by only one element in the domain.

Function Tests in 2-Dimensions

Vertical Line Test – If a vertical line passes through the graph of a relation in any two locations, it is not a function. If it is not possible to construct a vertical line that passes through the graph of a relation in two locations, it is a function.

Horizontal Line Test – If a horizontal line passes through the graph of a function in any two locations, it is not a one-to-one function. If it is not possible to construct a horizontal line that passes through the graph of a function in two locations, it is a one-to-one function.

Examples:

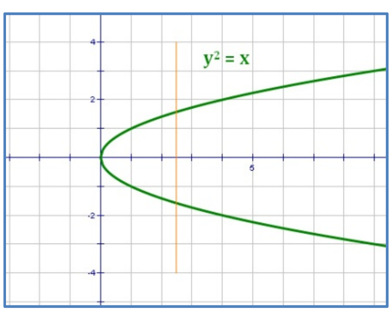


Figure 1: $y^2 = x$

Not a function.

Fails vertical line test.

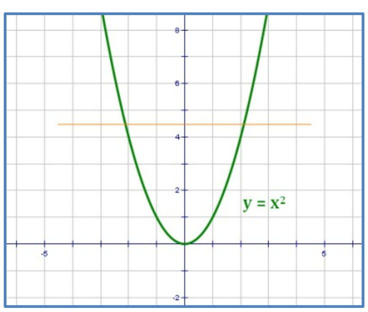


Figure 2: $y = x^2$

Is a function, but not a one-to-one function.

Passes vertical line test.

Fails horizontal line test.

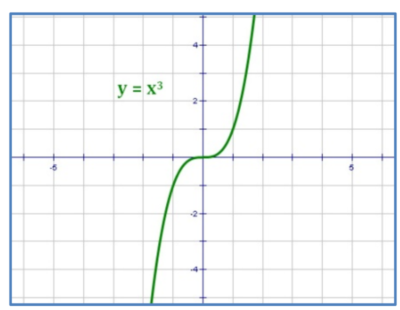


Figure 3: $y = x^3$

Is a one-to-one function.

Passes vertical line test.

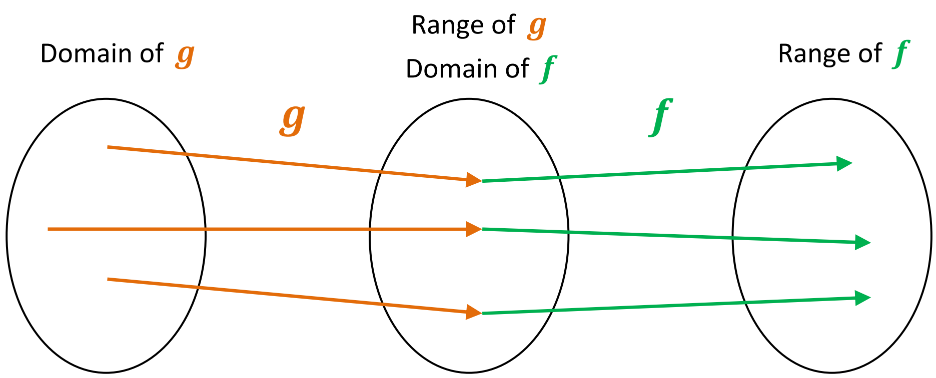
Passes horizontal line test.

Algebra
Composition of Functions

In a **Composition of Functions**, first one function is performed, and then the other. The notation for composition is, for example: $f(g(x))$ or $(f \circ g)(x)$. In both of these notations, the function g is performed first, and then the function f is performed on the result of g. Always perform the function closest to the variable first.

Double Mapping

A composition can be thought of as a double mapping. First g maps from its domain to its range. Then, f maps from the range of g to the range of f:



The Words Method

Example: Let	$f(x) = x^2$
and	$g(x) = x + 1$
Then:	$(f \circ g)(x) = (x + 1)^2$
And:	$(g \circ f)(x) = x^2 + 1$

In the example,

- The function f says *square the argument*.
- The function g says *add 1 to the argument*.

Sometimes it is easier to think of the functions in words rather than in terms of an argument like x.

$(f \circ g)$ says "add 1 first, then square the result."

$(g \circ f)$ says "square first, then add 1 to the result."

Using the words method,

Calculate:	$(f \circ g)(12yz)$
g: add 1 to it	$(12yz) + 1$
f: square it	$(12yz + 1)^2$

Calculate:	$(g \circ f)(-2)$
f: square it	$(-2)^2 = 4$
g: add 1 to it	$4 + 1 = 5$

Algebra
Inverses of Functions

In order for a function to have an inverse, it must be a one-to-one function. The requirement for a function to be an inverse is:

$$(\boldsymbol{f} \text{ o } \boldsymbol{f}^{-1})(\boldsymbol{x}) = (\boldsymbol{f}^{-1} \text{ o } \boldsymbol{f})(\boldsymbol{x}) = \boldsymbol{x}$$

The notation $\boldsymbol{f^{-1}(x)}$ is used for the **Inverse Function** of $\boldsymbol{f(x)}$.

Another way of saying this is that **if $\boldsymbol{f(a) = b}$, then $\boldsymbol{f^{-1}(b) = a}$ for all $\boldsymbol{a}$ in the domain of $\boldsymbol{f}$**.

Deriving an Inverse Function

The following steps can be used to derive an inverse function. This process assumes that the original function is expressed in terms of $f(x)$.

- Make sure the function is one-to-one. Otherwise it has no inverse. You can accomplish this by graphing the function and applying the vertical and horizontal line tests.
- Substitute the variable $\boldsymbol{y}$ for $\boldsymbol{f(x)}$.
- Exchange variables. That is, change all the $\boldsymbol{x}$'s to $\boldsymbol{y}$'s and all the $\boldsymbol{y}$'s to $\boldsymbol{x}$'s.
- Solve for the new $\boldsymbol{y}$ in terms of the new $\boldsymbol{x}$.
- (Optional) Switch the expressions on each side of the equation if you like.
- Replace the variable $\boldsymbol{y}$ with the function notation $\boldsymbol{f^{-1}(x)}$.
- Check your work.

Examples:

Derive the inverse of: $f(x) = \frac{1}{3}x + 2$

Substitute y for $f(x)$:	$y = \frac{1}{3}x + 2$
Exchange variables:	$x = \frac{1}{3}y + 2$
Subtract 2:	$x - 2 = \frac{1}{3}y$
Multiply by 3:	$3x - 6 = y$
Switch sides:	$y = 3x - 6$
Change Notation:	$\boldsymbol{f^{-1}(x) = 3x - 6}$

To check the result, note that:

$$(f^{-1} \text{ o } f)(x) = 3f(x) - 6 = 3\left(\frac{1}{3}x + 2\right) - 6 = x$$

Derive the inverse of: $f(x) = 2x - 1$

Substitute y for $f(x)$:	$y = 2x - 1$
Exchange variables:	$x = 2y - 1$
Add 1:	$x + 1 = 2y$
Divide by 2:	$\frac{x+1}{2} = y$
Switch sides:	$y = \frac{x+1}{2}$
Change Notation:	$\boldsymbol{f^{-1}(x) = \frac{x+1}{2}}$

To check the result, note that:

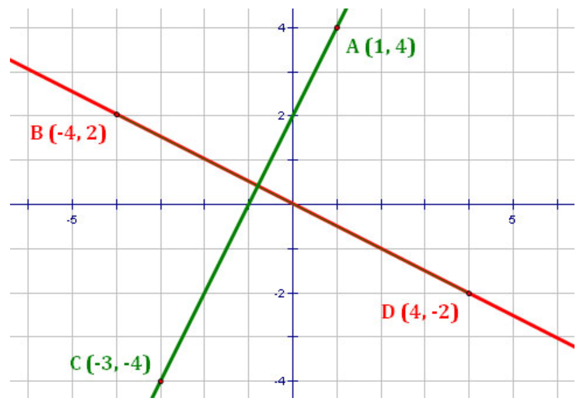
$$(f^{-1} \text{ o } f)(x) = \frac{f(x) + 1}{2} = \frac{(2x - 1) + 1}{2} = x$$

Algebra
Slope of a Line

The slope of a line tells how fast it rises or falls as it moves from left to right. If the slope is rising, the slope is positive; if it is falling, the slope is negative. The letter "***m***" is often used as the symbol for slope.

The two most useful ways to calculate the slope of a line are discussed below.

Mathematical Definition of Slope



The definition is based on two points with coordinates (x_1, y_1) and (x_2, y_2). The definition, then, is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Comments:

- You can select any 2 points on the line.
- A table such as the one at right can be helpful for doing your calculations.
- Note that $m = \frac{y_2 - y_1}{x_2 - x_1}$ implies that $m = \frac{y_1 - y_2}{x_1 - x_2}$. So, it does not matter which point you assign as Point 1 and which you assign as Point 2. Therefore, neither does it matter which point is first in the table.
- It is important that once you assign a point as Point 1 and another as Point 2, that you use their coordinates in the proper places in the formula.

	x-value	y-value
Point 2	x_2	y_2
Point 1	x_1	y_1
Difference	$x_2 - x_1$	$y_2 - y_1$

Examples:

For the two lines in the figure above, we get the following:

Green Line	**x-value**	**y-value**
Point A	1	4
Point C	-3	-4
Difference	4	8

Red Line	**x-value**	**y-value**
Point D	4	-2
Point B	-4	2
Difference	8	-4

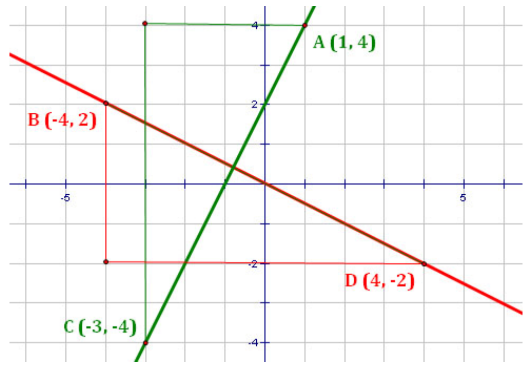
Green Line: $m = \frac{8}{4} = 2$

Red Line: $m = \frac{-4}{8} = -\frac{1}{2}$

Algebra
Slope of a Line (cont'd)

Rise over Run

An equivalent method of calculating slope that is more visual is the "Rise over Run" method. Under this method, it helps to draw vertical and horizontal lines that indicate the horizontal and vertical distances between points on the line.



The slope can then be calculated as follows:

$$m = \frac{rise}{run} = \frac{length\ of\ vertical\ line}{length\ of\ horizontal\ line}$$

The **rise** of a line is how much it **increases (positive)** or **decreases (negative)** between two points. The **run** is how far the line moves to the **right (positive)** or the **left (negative)** between the same two points.

Comments:

- You can select any 2 points on the line.
- It is important to start at the same point in measuring both the rise and the run.
- A good convention is to always start with the point on the left and work your way to the right; that way, the run (i.e., the denominator in the formula) is always positive. The only exception to this is when the run is zero, in which case the slope is undefined.
- If the two points are clearly marked as integers on a graph, the rise and run may actually be counted on the graph. This makes the process much simpler than using the formula for the definition of slope. However, when counting, make sure you get the right sign for the slope of the line, e.g., moving down as the line moves to the right is a negative slope.

Examples:

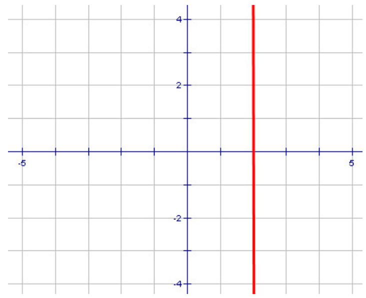
For the two lines in the figure above, we get the following:

Green Line: $m = \frac{rise\ from (-4)\ to\ 4}{run\ from\ (-3) to\ 1} = \frac{8}{4} = 2$

Red Line: $m = \frac{fall\ from\ 2\ to\ (-2)}{run\ from\ (-4)\ to\ 4} = \frac{-4}{8} = -\frac{1}{2}$

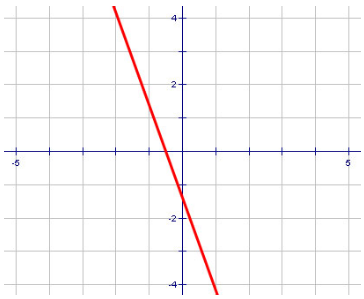
Notice how similar the calculations in the examples are under the two methods of calculating slopes.

Algebra
Slopes of Various Lines



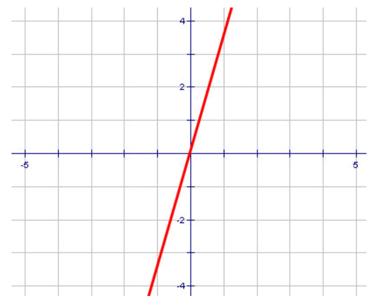
$m = undefined$

line is vertical



$m = -2\frac{4}{5}\ (big\ negative)$

line is steep and going down



$m = 3\frac{1}{2}\ (big\ positive)$

line is steep and going up

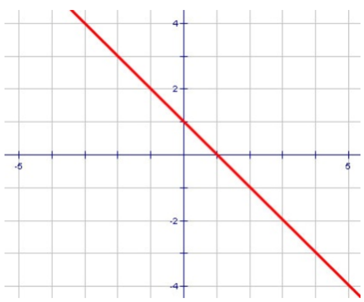
When you look at a line, you should notice the following about its slope:

- *Whether it is 0, positive, negative or undefined.*
- *If positive or negative, whether it is less than 1, about 1, or greater than 1.*

The purpose of the graphs on this page is to help you get a feel for these things.

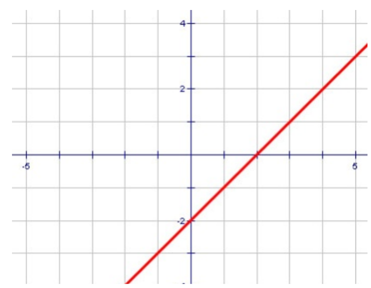
This can help you check:

- Given a slope, whether you drew the line correctly, or
- Given a line, whether you calculated the slope correctly.



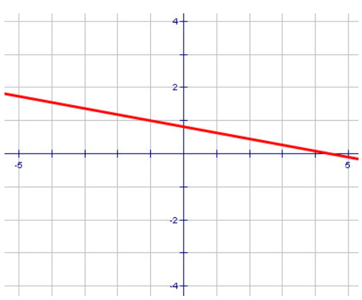
$m = -1$

line goes down at a 45^o angle



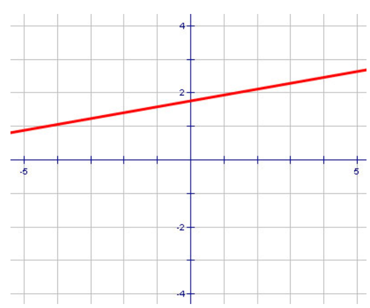
$m = 1$

line goes up at a 45^o angle



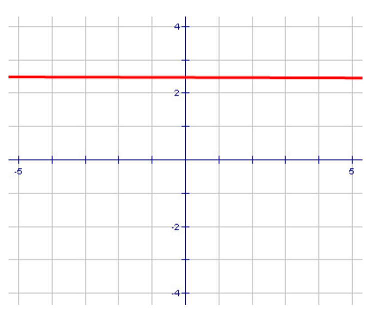
$m = -\frac{3}{17}\ (small\ negative)$

line is shallow and going down



$m = \frac{2}{11}\ (small\ positive)$

line is shallow and going up



$m = 0$

line is horizontal

Algebra
Various Forms of a Line

There are three forms of a linear equation which are most useful to the Algebra student, each of which can be converted into the other two through algebraic manipulation. The ability to move between forms is a very useful skill in Algebra, and should be practiced by the student.

Standard Form

The Standard Form of a linear equation is:

$$\boldsymbol{Ax + By = C}$$

where A, B, and C are real numbers and A and B are not both zero. Usually in this form, the convention is for A to be positive.

Standard Form Examples

$$3x + 2y = 6$$

$$2x - 7y = 14$$

Why, you might ask, is this "Standard Form?" One reason is that this form is easily extended to additional variables, whereas other forms are not. For example, in four variables, the Standard Form would be: $\boldsymbol{Ax + By + Cz + Dw = E}$. Another reason is that this form easily lends itself to analysis with matrices, which can be very useful in solving systems of equations.

Slope-Intercept Form

The Slope-Intercept Form of a linear equation is the one most familiar to many students. It is:

$$\boldsymbol{y = mx + b}$$

where m is the slope and b is the y-intercept of the line (i.e., the value at which the line crosses the y-axis in a graph). m and b must also be real numbers.

Slope-Intercept Examples

$$y = -3x + 6$$

$$y = \frac{3}{4}x - 14$$

Point-Slope Form

The Point-Slope Form of a linear equation is the one used least by the student, but it can be very useful in certain circumstances. In particular, as you might expect, it is useful if the student is asked for the equation of a line and is given the line's slope and the coordinates of a point on the line. The form of the equation is:

$$\boldsymbol{(y - y_1) = m(x - x_1)}$$

where m is the slope and (x_1, y_1) is any point on the line. One strength of this form is that **equations formed using different points on the same line will be equivalent.**

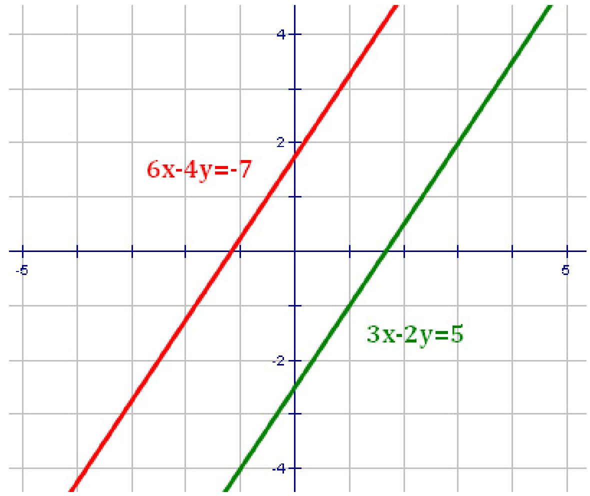
Point-Slope Examples

$$(y - 3) = 2\,(x + 4)$$

$$(y + 7) = 5\left(x - \frac{2}{3}\right)$$

Algebra
Slopes of Parallel and Perpendicular Lines

Parallel Lines



Two lines are parallel if their slopes are equal.

- In $y = mx + b$ form, if the values of m are the same.

  Example: $y = 2x - 3$ and
  $y = 2x + 1$

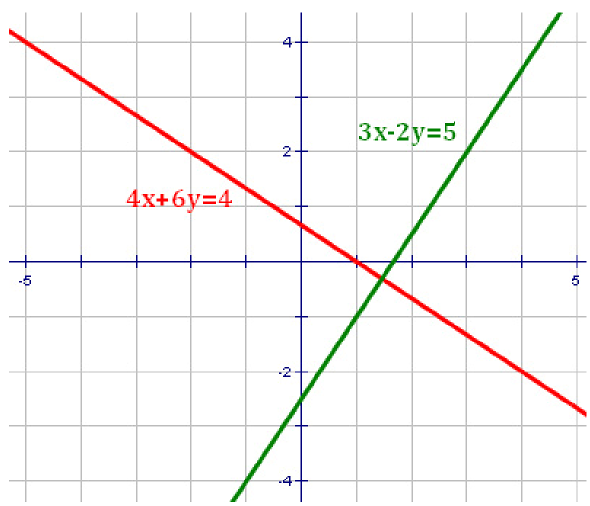
- In Standard Form, if the coefficients of x and y are proportional between the equations.

  Example: $3x - 2y = 5$ and
  $6x - 4y = -7$

- Also, if the lines are both vertical (i.e., their slopes are undefined).

  Example: $x = -3$ and
  $x = 2$

Perpendicular Lines



Two lines are perpendicular if the product of their slopes is -1. That is, if the slopes have different signs and are multiplicative inverses.

- In $y = mx + b$ form, the values of m multiply to get -1..

  Example: $y = 6x + 5$ and
  $y = -\frac{1}{6}x - 3$

- In Standard Form, if you add the product of the x-coefficients to the product of the y-coefficients and get zero.

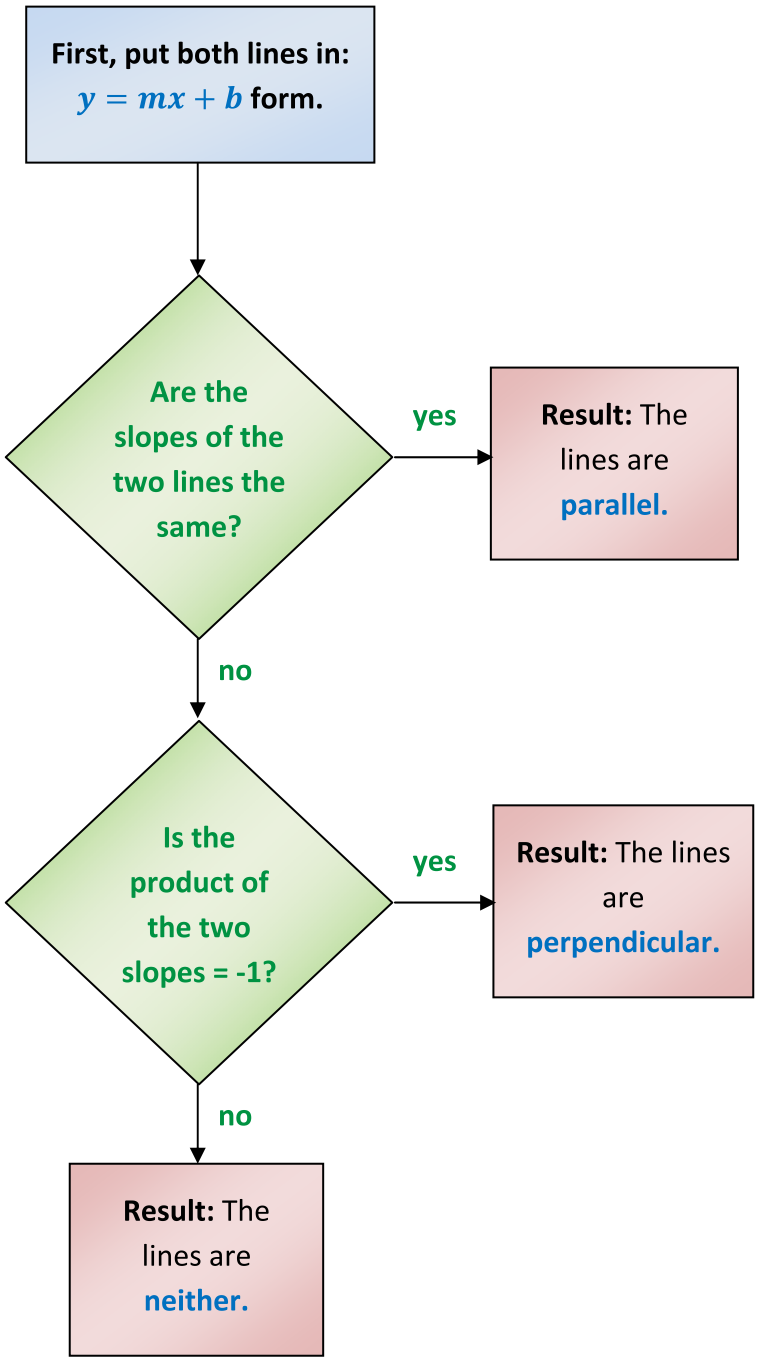
  Example: $4x + 6y = 4$ and
  $3x - 2y = 5$ because $(4 \cdot 3) + \left(6 \cdot (-2)\right) = 0$

- Also, if one line is vertical (i.e., m is undefined) and one line is horizontal (i.e., $m = 0$).

  Example: $x = 6$ and
  $y = 3$

Algebra
Parallel, Perpendicular or Neither

The following flow chart can be used to determine whether a pair of lines are parallel, perpendicular, or neither.



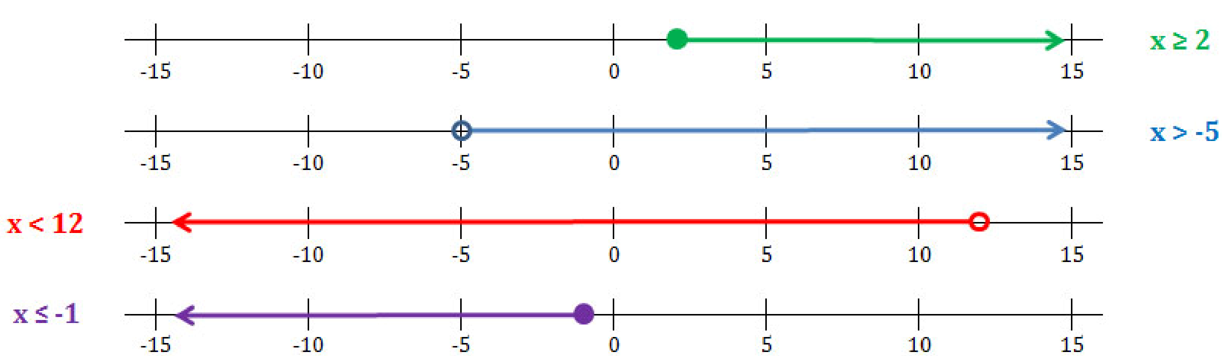
Algebra
Graphs of Inequalities in One Dimension

Inequalities in one dimension are generally graphed on the number line. Alternatively, if it is clear that the graph is one-dimensional, the graphs can be shown in relation to a number line but not specifically on it (examples of this are on the next page).

One-Dimensional Graph Components

- **The endpoint(s)** – The endpoints for the ray or segment in the graph are shown as either open or closed circles.
    - If the point is included in the solution to the inequality (i.e., if the sign is ≤ or ≥), the circle is closed.
    - If the point is not included in the solution to the inequality (i.e., if the sign is < or >), the circle is open.
- **The arrow** – If all numbers in one direction of the number line are solutions to the inequality, an arrow points in that direction.
    - For < or ≤ signs, the arrow points to the left (←).
    - For > or ≥ signs, the arrow points to the right (→).
- **The line** – in a simple inequality, a line is drawn from the endpoint to the arrow. If there are two endpoints, a line is drawn from one to the other.

Examples:



Algebra
Compound Inequalities in One Dimension

Compound inequalities are a set of inequalities that must all be true at the same time. Usually, there are two inequalities, but more than two can also form a compound set. The principles described below easily extend to cases where there are more than two inequalities.

Compound Inequalities with the Word "AND"

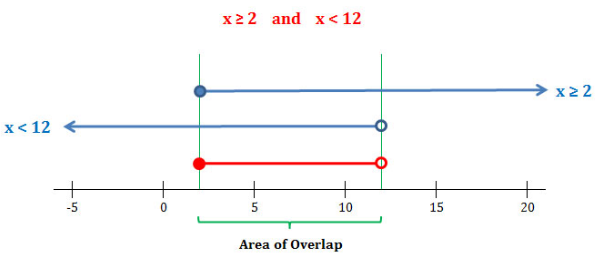
An example of compound inequalities with the word "AND" would be:

$x < 12 \quad and \quad x \geq 2$ **or** $2 \leq x < 12$

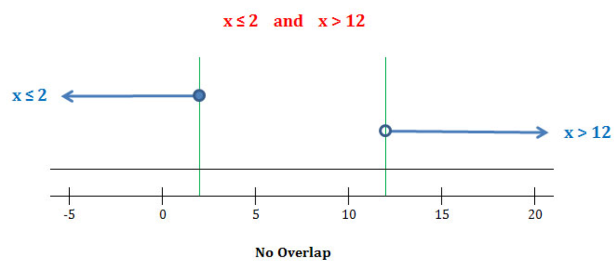
(Simple Form) **(Compound Form)**

These are the same conditions, expressed in two different forms.

Graphically, "AND" inequalities exist at points where the graphs of the individual inequalities overlap. This is the "intersection" of the graphs of the individual inequalities. Below are two examples of graphs of compound inequalities using the word "AND."



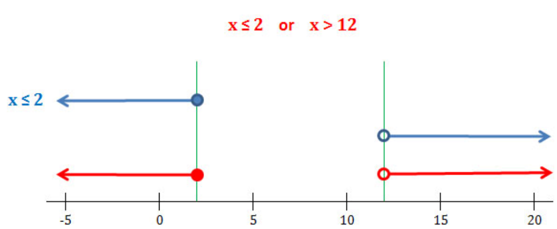
A typical "AND" example: The result is a segment that contains the points that overlap the graphs of the individual inequalities.



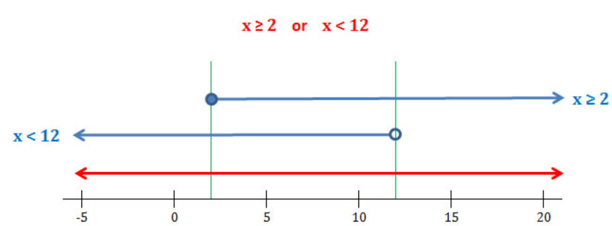
"AND" compound inequalities sometimes result in the empty set. This happens when no numbers meet both conditions at the same time.

Compound Inequalities with the Word "OR"

Graphically, "OR" inequalities exist at points where any of the original graphs have points. This is the "union" of the graphs of the individual inequalities. Below are two examples of graphs of compound inequalities using the word "OR."



A typical "OR" example: The result is a pair of rays extending in opposite directions, with a gap in between.



"OR" compound inequalities sometimes result in the set of all numbers. This happens when every number meets at least one of the conditions.

Algebra
Absolute Value Functions

Equations

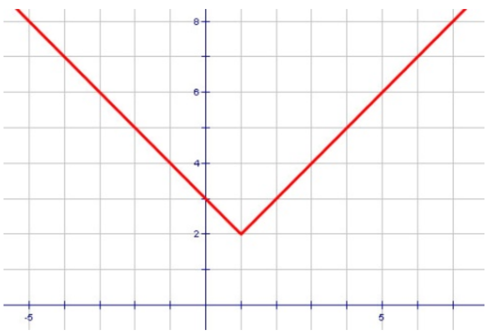
Graphs of equations involving absolute values generally have a "V" pattern. Whenever you see a "V" in a graph, think "absolute value." A general equation for an absolute value function is of the form:

$$y = |m(x-h)| + k \qquad or \qquad y = -|m(x-h)| + k$$

where,

- the sign indicates whether the graph opens up ("+" sign) or down ("−" sign).
- $|m|$ is the absolute value of the slopes of the lines in the graph.
- (h, k) is the location of the vertex (i.e., the sharp point) in the graph.

Examples:

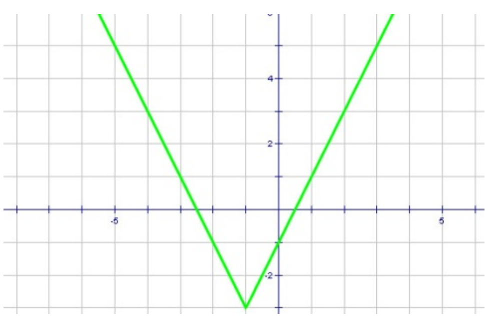


Equation: $y = |x-1| + 2$

Vertex = $(1, 2)$

$m = 1$; $|\text{slopes}| = 1$

Graph opens up

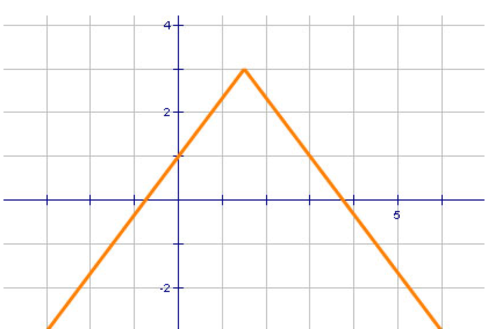


Equation: $y = |-2(x+1)| - 3$

Vertex = $(-1, -3)$

$m = -2$; $|\text{slopes}| = 2$

Graph opens up



Equation: $y = -\left|\frac{4}{3}\left(x - \frac{3}{2}\right)\right| + 3$

Vertex = $\left(\frac{3}{2}, 3\right)$

$m = \frac{4}{3}$; $|\text{slopes}| = \frac{4}{3}$

Graph opens down

Algebra
Absolute Value Functions (cont'd)

Inequalities

Since a positive number and a negative number can have the same absolute value, inequalities involving absolute values must be broken into two separate equations. For example:

$$|x-3| < 4 \quad \rightarrow \quad x-3<4 \quad \text{and} \quad x-3>(-4)$$

Sign that determines use of "AND" or "OR"

The first new equation is simply the original equation without the absolute value sign.

In the second new equation, two things change: (1) the sign flips, and (2) the value on the right side of the inequality changes its sign.

At this point the absolute value problem has converted into a pair of compound inequalities.

Equation 1

Solve:	$x-3 < 4$
Step 1: Add 3	$+3 \quad +3$
Result:	$x < 7$

Equation 2

Solve:	$x-3 > -4$
Step 1: Add 3	$+3 \quad +3$
Result:	$x > -1$

Next, we need to know whether to use "AND" or "OR" with the results. To decide which word to use, look at the sign in the inequality; then …

- Use the word **"AND"** with **"less th<u>and</u>"** signs.
- Use the word **"OR"** with **"great<u>or</u>"** signs.

Note: the English is poor, but the math is easier to remember with this trick!

The solution to the above absolute value problem, then, is the same as the solution to the following set of compound inequalities:

$x<7 \quad and \quad x>(-1)$ The solution set is all x in the range (-1, 7)

Note: the solution set to this example is given in **"range" notation**. When using this notation,

- use parentheses ( ) whenever an endpoint is not included in the solution set, and
- use square brackets [ ] whenever an endpoint is included in the solution set.
- Always use parentheses ( ) with infinity signs ($-\infty$ *or* ∞).

Examples:

The range: $x<6$ *and* $x \geq 2$
Notation: $[2, 6)$

The range: $x \leq -2$
Notation: $(-\infty, -2]$

Algebra
Systems of Equations

A system of equations is a set of 2 or more equations for which we wish to determine all solutions which satisfy each equation. Generally, there will be the same number of equations as variables and a single solution to each variable will be sought. However, sometimes there is either no solution or there is an infinite number of solutions.

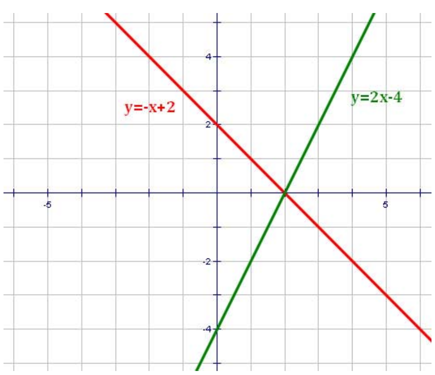
There are many methods available to solve a system of equations. We will show three of them below.

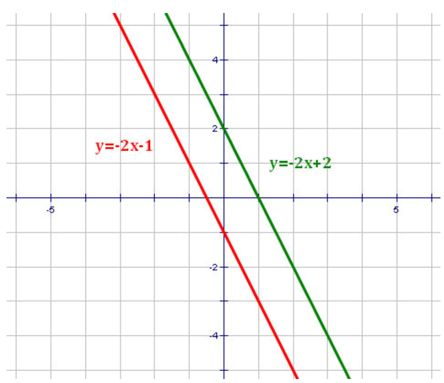
Graphing a Solution

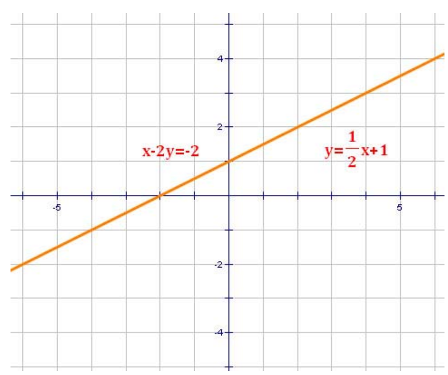
In the simplest cases, a set of 2 equations in 2 unknowns can be solved using a graph. A single equation in two unknowns is a line, so two equations give us 2 lines. The following situations are possible with 2 lines:

- **They will intersect.** In this case, the point of intersection is the only solution.
- **They will be the same line.** In this case, all points on the line are solutions (note: this is an infinite set).
- **They will be parallel but not the same line.** In this case, there are no solutions.

Examples







Solution Set:

The point of intersection can be read off the graph; the point (2,0).

Solution Set:

The empty set; these parallel lines will never cross.

Solution Set:

All points on the line. Although the equations look different, they actually describe the same line.

Algebra
Systems of Equations (cont'd)

Substitution Method

In the **Substitution Method**, we eliminate one of the variables by substituting into one of the equations its equivalent in terms of the other variable. Then we solve for each variable in turn and check the result. The steps in this process are illustrated in the example below.

Example: Solve for x and y if: $y = -x + 2$

and: $2x = 3y + 9.$

Step 1: Review the two equations. Look for a variable that can be substituted from one equation into the other. In this example, we see a single ***"y"*** in the first equation; this is a prime candidate for substitution.

We will substitute $(-x + 2)$ from the first equation for y in the second equation.

Step 2: Perform the substitution.

$2x = 3y + 9$ becomes: $2x = 3(-x + 2) + 9$

Step 3: Solve the resulting equation for the single variable that is left.

$$2x = 3(-x + 2) + 9$$
$$2x = -3x + 6 + 9$$
$$2x = -3x + 15$$

$$2x = -3x + 15$$
$$5x = 15$$
$$x = 3$$

Step 4: Substitute the known variable into one of the original equations to solve for the remaining variable.

$$y = -x + 2$$
$$y = -(3) + 2$$
$$y = -1$$

After this step, the solution is tentatively identified as: $x = 3, y = -1$, meaning the point **(3, -1)**.

Step 5: Check the result by substituting the solution into the equation not used in Step 4. If the solution is correct, the result should be a true statement. If it is not, you have made a mistake and should check your work carefully.

$$2x = 3y + 9$$
$$2(3) = 3(-1) + 9$$
$$6 = -3 + 9$$

Since this is a true mathematical statement, the solution **(3, -1)** can be accepted as correct.

Algebra
Systems of Equations (cont'd)

Elimination Method

In the **Substitution Method**, we manipulate one or both of the equations so that we can add them and eliminate one of the variables. Then we solve for each variable in turn and check the result. This is an outstanding method for systems of equations with "ugly" coefficients. The steps in this process are illustrated in the example below. Note the flow of the solution on the page.

Example: Solve for x and y if: $y = -x + 2$
and: $2x = 3y + 9.$

Step 1: Re-write the equations in standard form.

$$x + y = 2$$
$$2x - 3y = 9$$

Step 2: Multiply each equation by a value selected so that, when the equations are added, a variable will be eliminated.

$x + y = 2$ → (Multiply by 2) → $2x + 2y = 4$

$2x - 3y = 9$ → (Multiply by -1) → $-2x + 3y = -9$

Step 3: Add the resulting equations.

$$2x + 2y = 4$$
$$-2x + 3y = -9$$
$$5y = -5$$

Step 4: Solve for the variable.

$$5y = -5$$
$$y = -1$$

Step 5: Substitute the result into one of the original equations and solve for the other variable.

$$y = -x + 2$$
$$-1 = -x + 2$$
$$x - 1 = 2$$
$$x = 3$$

Step 6: Check the result by substituting the solution into the equation not used in Step 5. If the solution is correct, the result should be a true statement. If it is not, you have made a mistake and should check your work.

$$2x = 3y + 9$$
$$2(3) = 3(-1) + 9$$
$$6 = -3 + 9$$

Since this is a true mathematical statement, the solution **(3, -1)** can be accepted as correct.

Algebra
Systems of Equations (cont'd)

Classification of Systems

There are two main classifications of systems of equations: Consistent vs. Inconsistent, and Dependent vs. Independent.

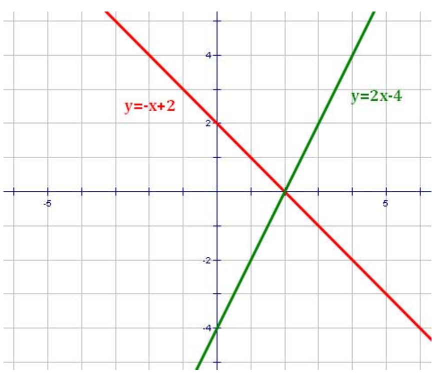
Consistent vs. Inconsistent

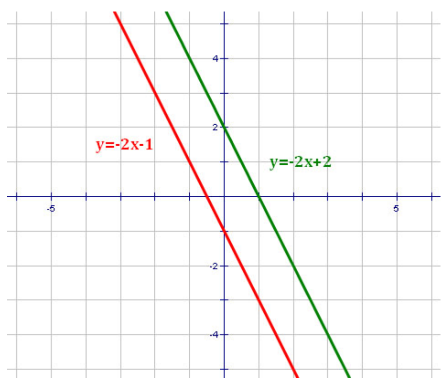
- **Consistent Systems** have one or more solutions.
- **Inconsistent Systems** have no solutions. When you try to solve an inconsistent set of equations, you often get to a point where you have an impossible statement, such as "$1 = 2$." This indicates that there is no solution to the system.

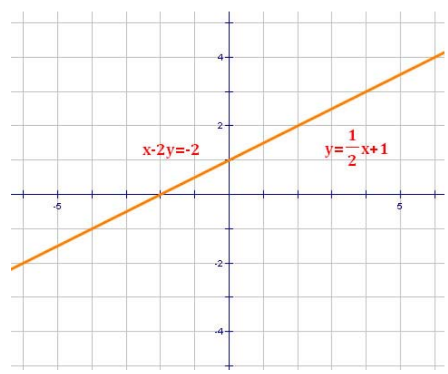
Dependent vs. Independent

- **Linearly Dependent Systems** have an infinite number of solutions. In **Linear Algebra**, a system is linearly dependent if there is a set of real numbers (not all zero) that, when they are multiplied by the equations in the system and the results are added, the final result is zero.
- **Linearly Independent Systems** have at most one solution. In **Linear Algebra**, a system is linearly independent if it is not linearly dependent. **Note:** some textbooks indicate that an independent system must have a solution. This is not correct; they can have no solutions (see the middle example below). *For more on this, see the next page.*

Examples







One Solution	No Solution	Infinite Solutions
Consistent	Inconsistent	Consistent
Independent	Independent	Dependent

Algebra
Adding and Subtracting Polynomials

Problems asking the student to add or subtract polynomials are often written in linear form:

$$\text{Add: } (3x^3 + 2x - 4) + (-2x^2 + 4x + 6)$$

The problem is much more easily solved if the problem is written in column form, with each polynomial written in standard form.

Definitions

Standard Form: A polynomial in standard form has its terms written from highest degree to lowest degree from left to right.

Example: The standard form of $(x + 3x^2 + 4)$ is $(3x^2 + x + 4)$

Like Terms: Terms with the same variables raised to the same powers. Only the numerical coefficients are different.

Example: $2xz^5$, $-6xz^5$, and xz^5 are like terms.

Addition and Subtraction Steps

Step 1: Write each polynomial in standard form. Leave blank spaces for missing terms. For example, if adding $(3x^3 + 2x - 4)$, leave space for the missing x^2-term.

Step 2: If you are subtracting, change the sign of each term of the polynomial to be subtracted and add instead. Adding is much easier than subtracting.

Step 3: Place the polynomials in column form, being careful to line up like terms.

Step 4: Add the polynomials.

Examples:

Add: $(3x^3 + 2x - 4) + (-2x^2 + 4x + 6)$

Solution:

$$\begin{array}{rrrrr} & 3x^3 & & +2x & -4 \\ + & & -2x^2 & +4x & +6 \\ \hline & 3x^3 & -2x^2 & +6x & +2 \end{array}$$

Subtract: $(3x^3 + 2x - 4) - (-2x^2 + 4x + 6)$

Solution:

$$\begin{array}{rrrrr} & 3x^3 & & +2x & -4 \\ + & & 2x^2 & -4x & -6 \\ \hline & 3x^3 & +2x^2 & -2x & -10 \end{array}$$

Algebra
Multiplying Binomials

The three methods shown below are equivalent. Use whichever one you like best.

FOIL Method

FOIL stands for First, Outside, Inside, Last. To multiply using the FOIL method, you make four separate multiplications and add the results.

Example: Multiply $(2x+3)\cdot(3x-4)$

First: $2x\cdot 3x = 6x^2$

Outside: $2x\cdot(-4) = -8x$

Inside: $3\cdot(3x) = 9x$

Last: $3\cdot(-4) = -12$

The result is obtained by adding the results of the 4 separate multiplications.

F O I L

$$(2x+3)\cdot(3x-4) = 6x^2 - 8x + 9x - 12$$
$$= 6x^2 + x - 12$$

Box Method

The Box Method is pretty much the same as the FOIL method, except for the presentation. In the box method, a 2x2 array of multiplications is created, the 4 multiplications are performed, and the results are added.

Example: Multiply $(2x+3)\cdot(3x-4)$

Multiply	3x	−4
2x	$6x^2$	$-8x$
+3	$9x$	-12

The result is obtained by adding the results of the 4 separate multiplications.

$$(2x+3)\cdot(3x-4) = 6x^2 - 8x + 9x - 12$$
$$= 6x^2 + x - 12$$

Stacked Polynomial Method

A third method is to multiply the binomials like you would multiply 2-digit numbers. The name comes from how the two polynomials are placed in a “stack” in preparation for multiplication.

Example: Multiply $(2x+3)\cdot(3x-4)$

$$\begin{array}{r} (2x+3) \\ \cdot\;\;(3x-4) \\ \hline -8x-12 \\ 6x^2+9x\quad\;\; \\ \hline 6x^2+\;x-12 \end{array}$$

Algebra
Multiplying Polynomials

If the polynomials to be multiplied contain more than two terms (i.e., they are larger than binomials), the FOIL Method will not work. Instead, either the Box Method or the Stacked Polynomial Method should be used. Notice that each of these methods is essentially a way to apply the distributive property of multiplication over addition.

The methods shown below are equivalent. Use whichever one you like best.

Box Method

The Box Method is the same for larger polynomials as it is for binomials, except the box is bigger. An array of multiplications is created; the multiplications are performed; and like terms are added.

Example: Multiply $(x^3 - 2x^2 + 2x + 3) \cdot (2x^2 - 3x - 4)$

Multiply	$\mathbf{2x^2}$	$\mathbf{-3x}$	$\mathbf{-4}$
$\mathbf{x^3}$	$2x^5$	$-3x^4$	$-4x^3$
$\mathbf{-2x^2}$	$-4x^4$	$+6x^3$	$+8x^2$
$\mathbf{+2x}$	$+4x^3$	$-6x^2$	$-8x$
$\mathbf{+3}$	$+6x^2$	$-9x$	-12

Results:

$$(x^3 - 2x^2 + 2x + 3) \cdot (2x^2 - 3x - 4)$$

$$= \quad 2x^5$$

$$-4x^4 - 3x^4$$

$$+4x^3 + 6x^3 - 4x^3$$

$$+6x^2 - 6x^2 + 8x^2$$

$$-9x - 8x$$

Stacked Polynomial Method

In the Stacked Polynomial Method, the polynomials are multiplied using the same technique to multiply multi-digit numbers One helpful tip is to place the smaller polynomial below the larger one in the stack.

Results:

$$
\begin{array}{r}
x^3 - 2x^2 + 2x + 3 \\
\cdot \quad 2x^2 - 3x - 4 \\
\hline
- 4x^3 + 8x^2 - 8x - 12 \\
-3x^4 + 6x^3 - 6x^2 - 9x \quad\quad \\
2x^5 - 4x^4 + 4x^3 + 6x^2 \quad\quad\quad\quad \\
\hline
2x^5 - 7x^4 + 6x^3 + 8x^2 - 17x - 12
\end{array}
$$

Algebra
Dividing Polynomials

Dividing polynomials is performed much like dividing large numbers long-hand.

Long Division Method

This process is best described by example:

Example: $(2x^3 + 5x^2 + x - 2) \div (x + 2)$

Step 1: Set up the division like a typical long hand division problem.

$$x + 2 \,\overline{\big)\, 2x^3 + 5x^2 + x - 2}$$

Step 2: Divide the leading term of the dividend by the leading term of the divisor. Place the result above the term of like degree of the dividend.

$(2x^3) \div x = 2x^2$

$$\begin{array}{r} 2x^2 \qquad\qquad\quad \\ x + 2 \,\overline{\big)\, 2x^3 + 5x^2 + x - 2} \end{array}$$

Step 3: Multiply the new term on top by the divisor and subtract from the dividend.

$$(2x^2)(x + 2) = 2x^3 + 4x^2$$

$$\begin{array}{r} 2x^2 \qquad\qquad\quad \\ x + 2 \,\overline{\big)\, 2x^3 + 5x^2 + x - 2} \\ \underline{2x^3 + 4x^2} \qquad\quad \\ x^2 + x - 2 \end{array}$$

Step 4: Repeat steps 2 and 3 on the remainder of the division until the problem is completed.

$$\begin{array}{r} 2x^2 + x - 1 \\ x + 2 \,\overline{\big)\, 2x^3 + 5x^2 + x - 2} \\ \underline{2x^3 + 4x^2} \qquad\quad \\ x^2 + x - 2 \\ \underline{x^2 + 2x} \qquad \\ -x - 2 \\ \underline{-x - 2} \\ 0 \end{array}$$

This process results in the final answer appearing above the dividend, so that:

$(2x^3 + 5x^2 + x - 2) \div (x + 2) = 2x^2 + x - 1$

Remainders

If there were a remainder, it would be appended to the result of the problem in the form of a fraction, just like when dividing integers. For example, in the problem above, if the remainder were 3, the fraction $\frac{3}{x+2}$ would be added to the result of the division. $(2x^3 + 5x^2 + x + 1) \div (x + 2) = 2x^2 + x - 1 + \frac{3}{x+2}$

Alternatives

This process can be tedious. Fortunately, there are better methods for dividing polynomials than long division. These include **Factoring**, which is discussed next and elsewhere in this Guide, and **Synthetic Division**, which is discussed in the chapter on Polynomials – Intermediate.

Algebra
Factoring Polynomials

Polynomials cannot be divided in the same way numbers can. In order to divide polynomials, it is often useful to factor them first. Factoring involves extracting simpler terms from the more complex polynomial.

Greatest Common Factor

The Greatest Common Factor of the terms of a polynomial is determined as follows:

Step 1: Find the Greatest Common Factor of the coefficients.

Step 2: Find the Greatest Common Factor for each variable. This is simply each variable taken to the lowest power that exists for that variable in any of the terms.

Step 3: Multiply the GCF of the coefficients by the GCF for each variable.

Example:

Find the GCF of $(18x^5y^6z + 42x^3y^7z^3 + 30x^8z^6)$

The GCF of the coefficients and each variable are shown in the box to the right. The GCF of the polynomial is the product of the four individual GCFs.

$$\text{GCF }(18, 42, 30) = 6$$
$$\text{GCF }(x^5, x^3, x^8) = x^3$$
$$\text{GCF }(y^6, y^7, 1) = 1$$
$$\text{GCF }(z,\ z^3, z^6) = z$$
$$\text{So, GCF (polynomial)} = 6x^3z$$

Factoring Steps

Step 1: Factor out of all terms the GCF of the polynomial.

Step 2: Factor out of the remaining polynomial any binomials that can be extracted.

Step 3: Factor out of the remaining polynomial any trinomials that can be extracted.

Step 4: Continue this process until no further simplification is possible.

Note: Typically only steps 1 and 2 are needed in high school algebra problems.

Examples:

Factor: $3x^4y - 18x^3y + 27x^2y$

$$= 3x^2y\,(x^2 - 6x + 9)$$
$$= 3x^2y\,(x-3)^2$$

The factoring of the blue trinomial (2nd line) into the square of a binomial is the result of recognizing the special form it represents. Special forms are shown on the next two pages.

Factor: $6x^3y^3 - 24xy^3$

$$= 6xy^3\,(x^2 - 4)$$
$$= 6xy^3\,(x+2)(x-2)$$

The factoring of the blue binomial (2nd line) into binomials of lower degree is the result of recognizing the special form it represents. Special forms are shown on the next two pages.

Algebra
Special Forms of Quadratic Functions

It is helpful to be able to recognize a couple special forms of quadratic functions. In particular, if you can recognize perfect squares and differences of squares, your work will become easier and more accurate.

Perfect Squares

Perfect squares are of the form: $\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$

$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$

Identification and Solution

The following steps allow the student to identify and solve a trinomial that is a perfect square:

Step 1: Notice the first term of the trinomial is a square. Take its square root.

Step 2: Notice the last term of the trinomial is a square. Take its square root.

Step 3: Multiply the results of the first 2 steps and double that product. If the result is the middle term of the trinomial, the expression is a perfect square.

Step 4: The binomial in the solution is the sum or difference of the square roots calculated in steps 1 and 2. The sign between the terms of the binomial is the sign of the middle term of the trinomial.

Example:

$$\boldsymbol{4x^2 - 12xy + 9y^2}$$

$$\boldsymbol{\sqrt{4x^2} = \pm 2x} \qquad \boldsymbol{\sqrt{9y^2} = \pm 3y}$$

Notice that the middle term is double the product of the two square roots ($\boldsymbol{2x}$ and $\boldsymbol{3y}$). This is a telltale sign that the expression is a perfect square.

Identify the trinomial as a perfect square:

- Take the square roots of the first and last terms. They are $2x$ and $3y$.
- Test the middle term. Multiply the roots from the previous step, then double the result: $(2x \cdot 3y) \cdot 2 = 12xy$. The result (with a "$-$" sign in front) is the middle term of the original trinomial. Therefore, the expression is a perfect square.

To express the trinomial as the square of a binomial:

- The square roots of the first and last terms ($2x$ and $3y$) make up the binomial we seek.
- We may choose the sign of the first term, so let's choose the "$+$" sign.
- Having chosen the "$+$" sign for the first term, the second term of the binomial takes the sign of the middle term of the original trinomial ("$-$"). Therefore, the result is:

$$\boldsymbol{4x^2 - 12xy + 9y^2 = (2x - 3y)^2}$$

Algebra
Special Forms of Quadratic Functions

Differences of Squares

Differences of squares are of the form: $\quad \boldsymbol{a^2 - b^2 = (a+b)\cdot(a-b)}$

These are much easier to recognize than the perfect squares because there is no middle term to consider. Notice why there is no middle term:

$$(a+b)\cdot(a-b) = a^2 \underbrace{+\,ab - ab}_{\text{these two terms cancel}} - b^2 = a^2 - b^2$$

Identification

To see if an expression is a difference of squares, you must answer "yes" to four questions:

1. Are there only two terms?
2. Is there a "$-$" sign between the two terms?
3. Is the first term a square? If so, take its square root.
4. Is the second term a square? If so, take its square root.

The solution is the product of a) the sum of the square roots in questions 3 and 4, and b) the difference of the square roots in steps 3 and 4.

Note: *A telltale sign of when an expression might be the difference of 2 squares is when the coefficients on the variables are squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, etc.*

Examples:

(1) $\quad 4x^2 - 25y^2 = (2x+5y)\cdot(2x-5y)$

(2) $\quad x^2 - 49 = (x+7)\cdot(x-7)$

(3) $\quad 81 - 9z^2 = (9+3z)\cdot(9-3z)$

(4) $\quad \dfrac{x^2}{9} - \dfrac{y^2}{16} = \left(\dfrac{x}{3} + \dfrac{y}{4}\right)\cdot\left(\dfrac{x}{3} - \dfrac{y}{4}\right)$

ADVANCED: Over the field of complex numbers, it is also possible to factor the <u>sum</u> of 2 squares:

$$a^2 + b^2 = (a+bi)\cdot(a-bi)$$

This is not possible over the field of real numbers.

Algebra
Factoring Trinomials – Simple Case Method

A common problem in Elementary Algebra is the factoring of a trinomial that is neither a perfect square nor a difference of squares.

Consider the simple case where the coefficient of x^2 is 1. The general form for this case is:

$$(x+p)\cdot(x+q) = x^2 + \underbrace{(p+q)}_{}x + \underbrace{(pq)}_{}$$

In order to simplify the illustration of factoring a polynomial where the coefficient of x^2 is 1, we will use the orange descriptors above for the components of the trinomial being factored.

Simple Case Method

Step 1: Set up parentheses for a pair of binomials. Put "x" in the left hand position of each binomial.

Step 2: Put **sign 1** in the middle position in the left binomial.

Step 3: Multiply **sign 1** and **sign 2** to get the sign for the right binomial. Remember:

$(+)\cdot(+)=(+)$ $\quad$ $(-)\cdot(-)=(+)$

$(+)\cdot(-)=(-)$ $\quad$ $(-)\cdot(+)=(-)$

Step 4: Find two numbers that:

(a) Multiply to get the **constant**, and

(b) Add to get the **coefficient of x**

Fill in:

$\_\_\_ \cdot \_\_\_ = \_\_\_$

$\_\_\_ + \_\_\_ = \_\_\_$

Step 5: Place the numbers in the binomials so that their signs match the signs from Steps 2 and 3. **This is the final answer.**

Step 6: Check your work by multiplying the two binomials to see if you get the original trinomial.

Example: Factor $x^2 - 3x - 28$

$= (x \quad\ )\cdot(x \quad\ )$

$= (x - \quad)\cdot(x \quad\ )$

$= (x - \quad)\cdot(x + \quad)$

The numbers we seek are 4 and -7 because:

$4\cdot(-7) = -28$, and

$4 - 7 = -3$

$= (x-7)\cdot(x+4)$

$(x-7)\cdot(x+4)$

$= x^2 + 4x - 7x - 28$

$= x^2 - 3x - 28$

✓

Algebra
Factoring Trinomials – AC Method

There are times when the simple method of factoring a trinomial is not sufficient. Primarily this occurs **when the coefficient of x^2 is not 1**. In this case, you may use the AC method presented here, or you may use either the brute force method or the quadratic formula method (described on the next couple of pages).

AC Method

The AC Method derives its name from the first step of the process, which is to multiply the values of "a" and "c" from the general form of the quadratic equation: $\boldsymbol{y = ax^2 + bx + c}$

Step 1: Multiply the values of "a" and "c".

Step 2: Find two numbers that:

(a) Multiply to get the **value of ac**, and

(b) Add to get the **coefficient of x**

Fill in:

__ · __ = __

__ + __ = __

Step 3: Split the middle term into two terms, with coefficients equal to the values found in Step 2. (Tip: if only one of the coefficients is negative, put that term first.)

Step 4: Group the terms into pairs.

Step 5: Factor each pair of terms.

Step 6: Use the distributive property to combine the multipliers of the common term. **This is the final answer.**

Step 7: Check your work by multiplying the two binomials to see if you get the original trinomial.

Example: Factor $6x^2 - x - 2$

$6x^2 - x - 2$

-12

$(-4) \cdot 3 = -12$

$(-4) + 3 = -1$

$6x^2 - 4x + 3x - 2$

$(6x^2 - 4x) + (3x - 2)$

$2x(3x - 2) + 1(3x - 2)$

$= \boldsymbol{(2x + 1) \cdot (3x - 2)}$

$(2x + 1) \cdot (3x - 2)$

$= 6x^2 - 4x + 3x - 2$

$= 6x^2 - x - 2$

✓

Algebra
Table of Powers and Roots

Square Root	Number	Square	Cube	4th Power
$\sqrt{1} = 1.000$	1	$1^2 = 1$	$1^3 = 1$	$1^4 = 1$
$\sqrt{2} = 1.414$	2	$2^2 = 4$	$2^3 = 8$	$2^4 = 16$
$\sqrt{3} = 1.732$	3	$3^2 = 9$	$3^3 = 27$	$3^4 = 81$
$\sqrt{4} = 2.000$	4	$4^2 = 16$	$4^3 = 64$	$4^4 = 256$
$\sqrt{5} = 2.236$	5	$5^2 = 25$	$5^3 = 125$	$5^4 = 625$
$\sqrt{6} = 2.449$	6	$6^2 = 36$	$6^3 = 216$	$6^4 = 1{,}296$
$\sqrt{7} = 2.646$	7	$7^2 = 49$	$7^3 = 343$	$7^4 = 2{,}401$
$\sqrt{8} = 2.828$	8	$8^2 = 64$	$8^3 = 512$	$8^4 = 4{,}096$
$\sqrt{9} = 3.000$	9	$9^2 = 81$	$9^3 = 729$	$9^4 = 6{,}561$
$\sqrt{10} = 3.162$	10	$10^2 = 100$	$10^3 = 1{,}000$	$10^4 = 10{,}000$
$\sqrt{11} = 3.317$	11	$11^2 = 121$	$11^3 = 1{,}331$	$11^4 = 14{,}641$
$\sqrt{12} = 3.464$	12	$12^2 = 144$	$12^3 = 1{,}728$	$12^4 = 20{,}736$
$\sqrt{13} = 3.606$	13	$13^2 = 169$	$13^3 = 2{,}197$	$13^4 = 28{,}561$
$\sqrt{14} = 3.742$	14	$14^2 = 196$	$14^3 = 2{,}744$	$14^4 = 38{,}416$
$\sqrt{15} = 3.873$	15	$15^2 = 225$	$15^3 = 3{,}375$	$15^4 = 50{,}625$
$\sqrt{16} = 4.000$	16	$16^2 = 256$	$16^3 = 4{,}096$	$16^4 = 65{,}536$
$\sqrt{17} = 4.123$	17	$17^2 = 289$	$17^3 = 4.913$	$17^4 = 83{,}521$
$\sqrt{18} = 4.243$	18	$18^2 = 324$	$18^3 = 5{,}832$	$18^4 = 104{,}976$
$\sqrt{19} = 4.359$	19	$19^2 = 361$	$19^3 = 6{,}859$	$19^4 = 130{,}321$
$\sqrt{20} = 4.472$	20	$20^2 = 400$	$20^3 = 8{,}000$	$20^4 = 160{,}000$
$\sqrt{21} = 4.583$	21	$21^2 = 441$	$21^3 = 9{,}261$	$21^4 = 194{,}481$
$\sqrt{22} = 4.690$	22	$22^2 = 484$	$22^3 = 10{,}648$	$22^4 = 234{,}256$
$\sqrt{23} = 4.796$	23	$23^2 = 529$	$23^3 = 12{,}167$	$23^4 = 279841$
$\sqrt{24} = 4.899$	24	$24^2 = 576$	$24^3 = 13{,}824$	$24^4 = 331{,}776$
$\sqrt{25} = 5.000$	25	$25^2 = 625$	$25^3 = 15{,}625$	$25^4 = 390{,}625$

Algebra
The Quadratic Formula

The **Quadratic Formula** is one of the first difficult math formulas that students are asked to memorize. Mastering the formula, though difficult, is full of rewards. By knowing why it works and what the various parts of the formula are, a student can generate a lot of knowledge in a short period of time.

For a quadratic function of the form: $y = ax^2 + bx + c$

The formula for the roots (i.e., where $y = 0$) is: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ← **Quadratic Formula**

How Many Real Roots?

The discriminant is the part under the radical: $b^2 - 4ac$

- If the discriminant is negative, the quadratic function has **0** real roots. This is because a negative number under the radical results in imaginary roots instead of real roots. In this case the graph the graph will not cross the x-axis. It will be either entirely above the x-axis or entirely below the x-axis, depending on the value of "a".
- If the discriminant is zero, the quadratic function has **1** real root. The square root of zero is zero, so the radical disappears and the only root is $x = \left(\frac{-b}{2a}\right)$. In this case, the graph will appear to bounce off the x-axis; it touches the x-axis at only one spot – the value of the root.
- If the discriminant is positive, the quadratic function has **2** real roots. This is because a real square root exists, and it must be added in the formula to get one root and subtracted to get the other root. In this case, the graph will cross the x-axis in two places, the values of the roots.

Where are the Vertex and Axis of Symmetry?

The x-coordinate of the vertex is also easily calculated from the quadratic formula because the vertex is halfway between the two roots. If we average the two roots, the $\pm$ portion of the formula disappears and the resulting x-value is $x = \left(\frac{-b}{2a}\right)$. The y-value of the vertex must still be calculated, but the x-value can be read directly out of the quadratic formula.

Also, once the x-value of the vertex is known, the equation for the axis of symmetry is also known. It is the vertical line containing the vertex: $x = \left(\frac{-b}{2a}\right)$.

Algebra
Radical Rules

Simple Rules Involving Radicals

General Radical Rule	Rule for Square Roots	Example
$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$	$\sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$
$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$	$\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}$

Note also that: $\sqrt[n]{a} = a^{(1/n)}$

e.g., $\sqrt{a} = a^{(1/2)}$, so the rules for exponents also apply for roots.

Rationalizing the Denominator

Mathematicians prefer to keep radicals out of the denominator. Here are two methods to accomplish this, depending on what's in the denominator.

Case 1: *Simple radical in the denominator.* Solution: multiply the beginning expression by a fraction which is the offending radical divided by itself.

Example: $$\frac{2+\sqrt{3}}{4\sqrt{5}} = \frac{2+\sqrt{3}}{4\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}+\sqrt{15}}{20}$$

Case 2: *Number and radical in the denominator.* Solution: multiply by the beginning expression by a fraction which is designed to eliminate the radical from the denominator. The numerator and denominator of the fraction are created by changing the sign between the number and the radical in the denominator.

Example: $$\frac{\sqrt{7}}{3-\sqrt{5}} = \frac{\sqrt{7}}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{3\sqrt{7}+\sqrt{35}}{9-5} = \frac{3\sqrt{7}+\sqrt{35}}{4}$$

Algebra
Simplifying Square Roots – Two Methods

Method 1: Extracting Squares

In this method, you pull squares out from under the radical. This is the quickest method if you are comfortable with what the squares are and with dividing them out of larger numbers.

$1^2 = 1$	$11^2 = 121$
$2^2 = 4$	$12^2 = 144$
$3^2 = 9$	$13^2 = 169$
$4^2 = 16$	$14^2 = 196$
$5^2 = 25$	$15^2 = 225$
$6^2 = 36$	$16^2 = 256$
$7^2 = 49$	$17^2 = 289$
$8^2 = 64$	$18^2 = 324$
$9^2 = 81$	$19^2 = 361$
$10^2 = 100$	$20^2 = 400$

Examples: (1) $\sqrt{98} = \sqrt{49} \cdot \sqrt{2}$
$= 7\sqrt{2}$

(2) $\sqrt{9600} = \sqrt{100} \cdot \sqrt{96}$
$= \sqrt{100} \cdot \sqrt{16} \cdot \sqrt{6}$
$= 10 \cdot 4 \cdot \sqrt{6}$
$= 40\sqrt{6}$

Method 2: Extracting Prime Numbers

If you are not comfortable with Method 1, you can pull prime numbers out from under the radical and pair them up to simplify the square root.

Example: $\sqrt{54} = \sqrt{2} \cdot \sqrt{27}$
$= \sqrt{2} \cdot \sqrt{3} \cdot \sqrt{9}$
$= \sqrt{2} \cdot \sqrt{3} \cdot \sqrt{3} \cdot \sqrt{3}$
$= \sqrt{2} \cdot (\sqrt{3} \cdot \sqrt{3}) \cdot \sqrt{3}$
$= \sqrt{2} \cdot (3) \cdot \sqrt{3}$
$= 3 \cdot \sqrt{2} \cdot \sqrt{3}$
$= 3 \cdot \sqrt{6}$

Note that the last step is to re-combine roots that do not come in pairs.

> Method 2 may take a lot longer than Method 1, but it works. A good use for Method 2 is when you try using the quicker Method 1 but get stuck – then working with primes can get you back on track toward solving the problem.

Algebra
Solving Radical Equations

When an equation involves radicals, the radicals must be eliminated in order to obtain a solution. The one special thing about these equations is that, in the process of eliminating the radical, it is possible to add another solution that is not a solution to the original problem.

Solutions that are added by the process used to solve the problem are called **Extraneous Solutions**. At the end of the problem, we must check for extraneous solutions and eliminate them.

Solving a Radical Equation

The steps to solving an equation involving radicals are:

- Isolate the radical on one side of the equation. To do this, add or subtract any variables or constants that are on the same side of the equation as the radical.
- If the radical is a square root, square both sides of the equation. If the radical is a cube root, cube both sides, etc. This should get rid of the radical.
- If there are any radicals remaining in the problem, repeat the first two steps until they are gone.
- Solve the equation that remains.
- Check all solutions to the problem using the equation in the original statement of the problem.
- Discard extraneous roots.

Example: Solve $\sqrt{4x+5} = x$

Starting Problem:	$\sqrt{2x+6}+1 = x$
Subtract **1** from both sides:	$\sqrt{2x+6} = x-1$
Square both sides:	$2x+6 = x^2-2x+1$
Subtract $\mathbf{2x+6}$ from both sides:	$x^2-4x-5=0$
Factor:	$(x-5)(x+1)=0$
Obtain Preliminary Solutions:	$x = \{-1, 5\}$
Test $\mathbf{-1}$ as a solution:	$\sqrt{2(-1)+6}+1 = -1$?
Test **5** as a solution:	$\sqrt{4(5)+5} = 5$? ✓
Identify the final Solution Set:	$\mathbf{x=5}$

If we allowed $\sqrt{\mathbf{2(-1)+6}}$ to be **-2**, the equation would work and **-1** would work as a solution. However, the square root of a number is defined to be the positive root only. **So, -1 fails as a solution to the problem.**

Matrices
Addition and Scalar Multiplication

What is a Matrix?

A matrix is an ordered set of numbers set up in a 2-dimensional array. Matrices are very useful in algebra, statistics and other applications because they provide a concise way to carry out more complex mathematical methods and processes.

Matrices have dimensions, expressed as the number of rows x the number of columns. For example, a 2x3 matrix (read "2 by 3 matrix") has 2 rows and 3 columns. Knowing the dimensions of a matrix is important because many matrix operations can only occur on matrices with certain dimensions.

Adding Matrices

Each number in a matrix is called an element. Matrices are added by adding the corresponding elements in the matrices. Matrices must have the same dimensions to be added.

Example:

$$\begin{bmatrix} 2 & -3 & 1 \\ 5 & 1 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 2 & -2 \end{bmatrix}$$

1st row, 1st column: $2 + (-1) = 1$
1st row, 2nd column: $(-3) + 2 = -1$

Scalar Multiplication

Multiplying a matrix by a scalar (i.e., a number) is accomplished by multiplying each element in the matrix by the scalar. The term scalar simply refers to "scaling" the matrix by making its values larger or smaller. Scalar multiplication can be performed on matrices of any dimensions.

Example:

$$3 \cdot \begin{bmatrix} -1 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 12 \\ -6 & 3 & 0 \end{bmatrix}$$

1st row, 1st column: $3 \cdot (-1) = -3$
1st row, 2nd column: $3 \cdot 2 = 6$

Matrices
Multiplying Matrices

Multiplying Matrices

Multiplication of matrices is a more complex process. Although the student may find it difficult at first, it is a powerful tool that is useful in many fields of mathematics and science.

Matrix multiplication can be performed only on matrices that are **conformable** (i.e., compatible in size). In order for two matrices to be multiplied together, the number of columns in the first matrix must equal the number of rows in the second matrix. If an ***m x n*** matrix is multiplied by an ***n x p*** matrix, the result is an ***m x p*** matrix. This is illustrated as follows:

must match

$$[m\ x\ n] \quad \cdot \quad [n\ x\ p] \quad = \quad [m\ x\ p]$$

size of resulting matrix

To multiply matrices, you multiply the elements in a row of one matrix by the corresponding elements in a column of the other matrix and add the results. If row *i* in the first matrix is multiplied by row *j* in the second matrix, the result is placed in row *i*, column *j* of the resulting matrix. The element in position *i, j* of a matrix is often denoted $\boldsymbol{a_{i,j}}$.

Example 1:

$$\begin{bmatrix} 2 & -3 & 1 \\ 5 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -13 \end{bmatrix}$$

Notice that multiplying a ***2 x 3*** matrix by a ***3 x 2*** matrix results in a ***2 x 2*** matrix.

1st row, 1st column: [2 · 1] + [(-3) · 2] + [1 · 3] = -1
1st row, 2nd column: [2 · (-2)] + [(-3) · (-1)] + [1 · 1] = 0
2nd row, 1st column: [5 · 1] + [1 · 2] + [(-2) · 3] = 1
2nd row, 2nd column: [5 · (-2)] + [1 · (-1)] + [(-2) · 1] = -13

Example 2:

$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 1 \\ 5 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -8 & -5 & 5 \\ -1 & -7 & 4 \\ 11 & -8 & 1 \end{bmatrix}$$

Notice that multiplying a ***3 x 2*** matrix by a ***2 x 3*** matrix results in a ***3 x 3*** matrix.

From the examples, it is clear that **matrix multiplication is not commutative**. That is, if we name two matrices ***A*** and ***B***, **it is not necessarily true that *A·B = B·A***. Further, if matrices are not square (i.e., having the same number of rows and columns), matrix multiplication is never commutative; that is ***A·B ≠ B·A***.

Algebra
Exponent Formulas

Word Description of Property	Math Description of Property	Limitations on variables	Examples
Product of Powers	$a^p \cdot a^q = a^{(p+q)}$		$x^4 \cdot x^3 = x^7$ $x^5 \cdot x^{-8} = x^{-3}$
Quotient of Powers	$\frac{a^p}{a^q} = a^{(p-q)}$	$a \neq 0$	$\frac{y^5}{y^2} = y^3$
Power of a Power	$(a^p)^q = a^{(p \cdot q)}$		$(z^4)^3 = z^{12}$ $(x^{-3})^{-5} = x^{15}$
Anything to the zero power is 1	$a^0 = 1$	$a \neq 0$	$91^0 = 1$ $(xyz^3)^0 = 1,$ if $x, y, z \neq 0$
Negative powers generate the reciprocal of what a positive power generates	$a^{(-p)} = \frac{1}{a^p}$	$a \neq 0$	$x^{(-3)} = \frac{1}{x^3}$ $\left(\frac{1}{x}\right)^{-5} = x^5$
Power of a product	$(a \cdot b)^p = a^p \cdot b^p$		$(3y)^3 = 27y^3$ $[(x+1)z]^4 = (x+1)^4 z^4$
Power of a quotient	$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$	$b \neq 0$	$\left(\frac{x}{4}\right)^3 = \frac{x^3}{64}$
Converting a root to a power	$\sqrt[n]{a} = a^{(1/n)}$	$n \neq 0$	$\sqrt{x} = x^{1/2}$

Algebra
Logarithm Formulas

Word Description of Property	Math Description of Property	Limitations on variables	Examples
Definition of logarithm	$(\log_b x = a)$ implies $(b^a = x)$	$b, x > 0$ $b \neq 1$	$\log_3 x = 4$ implies $3^4 = x$ $\log_7(-49)$ is undefined
Log (base anything) of 1 is zero	$\log_b 1 = 0$	$b > 0$ $b \neq 1$	$\log_{32} 1 = 0$ $\ln 1 = 0$
Exponents and logs are inverse operators, leaving what you started with	$b^{(\log_b x)} = x$	$b, x > 0$ $b \neq 1$	$3^{(\log_3 92)} = 92$ $e^{(\ln x)} = x$
Logs and exponents are inverse operators, leaving what you started with	$\log_b(b^x) = x$	$b, x > 0$ $b \neq 1$	$\log_6(6^{xyz}) = xyz$ $\ln\,(e^{4y}) = 4y$
The log of a product is the sum of the logs	$\log_b(m \cdot n) = \log_b m + \log_b n$	$m, n, b > 0$ $b \neq 1$	$\log_2(32x) = 5 + \log_2 x$ $\ln\,(8e) = \ln(8) + 1$
The log of a quotient is the difference of the logs	$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$	$m, n, b > 0$ $b \neq 1$	$\log_3\left(\frac{3}{x}\right) = 1 - \log_3 x$ $\ln\left(\frac{12}{e}\right) = \ln(12) - 1$
The log of something to a power is the power times the log	$\log_b(m^p) = p \cdot \log_b m$	$m, b > 0$ $b \neq 1$	$\log_4(x^{23}) = 23 \cdot \log_4 x$ $\ln\,(3^z) = z \cdot \ln(3)$
Change the base to whatever you want by dividing by the log of the old base	$\log_b m = \frac{\log_a m}{\log_a b}$	$m, a, b > 0$ $a, b \neq 1$	$\log_{100} x = \frac{\log_{10} x}{2}$

Algebra
Table of Exponents and Logarithms

Definition: $b^a = c$ if and only if $\log_b c = a$

$2^0 = 1$	$\log_2 1 = 0$
$2^1 = 2$	$\log_2 2 = 1$
$2^2 = 4$	$\log_2 4 = 2$
$2^3 = 8$	$\log_2 8 = 3$
$2^4 = 16$	$\log_2 16 = 4$
$2^5 = 32$	$\log_2 32 = 5$
$2^6 = 64$	$\log_2 64 = 6$
$2^7 = 128$	$\log_2 128 = 7$
$2^8 = 256$	$\log_2 256 = 8$
$2^9 = 512$	$\log_2 512 = 9$
$2^{10} = 1024$	$\log_2 1024 = 10$
$3^0 = 1$	$\log_3 1 = 0$
$3^1 = 3$	$\log_3 3 = 1$
$3^2 = 9$	$\log_3 9 = 2$
$3^3 = 27$	$\log_3 27 = 3$
$3^4 = 81$	$\log_3 81 = 4$
$3^5 = 243$	$\log_3 243 = 5$
$4^0 = 1$	$\log_4 1 = 0$
$4^1 = 4$	$\log_4 4 = 1$
$4^2 = 16$	$\log_4 16 = 2$
$4^3 = 64$	$\log_4 64 = 3$
$4^4 = 256$	$\log_4 256 = 4$
$5^0 = 1$	$\log_5 1 = 0$
$5^1 = 5$	$\log_5 5 = 1$
$5^2 = 25$	$\log_5 25 = 2$
$5^3 = 125$	$\log_5 125 = 3$
$5^4 = 625$	$\log_5 625 = 4$

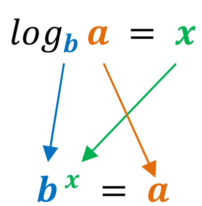
$6^0 = 1$	$\log_6 1 = 0$
$6^1 = 6$	$\log_6 6 = 1$
$6^2 = 36$	$\log_6 36 = 2$
$6^3 = 216$	$\log_6 216 = 3$
$7^0 = 1$	$\log_7 1 = 0$
$7^1 = 7$	$\log_7 7 = 1$
$7^2 = 49$	$\log_7 49 = 2$
$7^3 = 343$	$\log_7 343 = 3$
$8^0 = 1$	$\log_8 1 = 0$
$8^1 = 8$	$\log_8 8 = 1$
$8^2 = 64$	$\log_8 64 = 2$
$8^3 = 512$	$\log_8 512 = 3$
$9^0 = 1$	$\log_9 1 = 0$
$9^1 = 9$	$\log_9 9 = 1$
$9^2 = 81$	$\log_9 81 = 2$
$9^3 = 729$	$\log_9 729 = 3$
$10^0 = 1$	$\log_{10} 1 = 0$
$10^1 = 10$	$\log_{10} 10 = 1$
$10^2 = 100$	$\log_{10} 100 = 2$
$10^3 = 1000$	$\log_{10} 1000 = 3$
$11^0 = 1$	$\log_{11} 1 = 0$
$11^1 = 11$	$\log_{11} 11 = 1$
$11^2 = 121$	$\log_{11} 121 = 2$
$11^3 = 1331$	$\log_{11} 1331 = 3$

Algebra
Converting Between Exponential and Logarithmic Forms

To convert between an exponential expression and a logarithmic expression, it is often helpful to use the "first-last-middle" rule to perform the conversion. If necessasy, set the expression equal to x before applying the rule.

Note: the "first-last-middle" rule requires that the logarithmic or exponential portion of the expression be on the <u>*left-hand side*</u> *of the equation.*

Converting from Logarithmic Form to Exponential Form



using "**first**-**last**-**middle**"

Examples:

1) Solve for x: $log_4 64 = x$.

   First is "4", **last** is "x" and **middle** is "64." So, $4^x = 64.$

   Then, $4^1 = 4;\ 4^2 = 16;\ 4^3 = 64$ ✓

   So, we have: $\boldsymbol{x = 3}$

2) Solve for x: $\ln e = x$

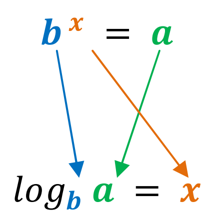
   (remember ln is shorthand for log_e)

   Using first-last-middle,

   $log_e\, e = x$ converts to: $e^x = e$

   So, we have: $\boldsymbol{x = 1}$

Converting from Exponential Form to Logarithmic Form



using "**first**-**last**-**middle**"

Examples:

1) Convert the expression, $2^5 = 32$ to logarithmic form.

   First is "2", **last** is "32" and **middle** is "5".

   So, we have: $\boldsymbol{log_2 32 = 5}$

2) Convert the expression, $7^3 = 343$ to logarithmic form.

   Using first-last-middle,

   $7^3 = 343$ converts to: $log_7 343 = 3$

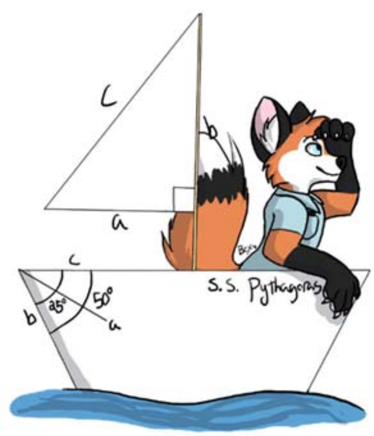
   So, we have: $\boldsymbol{log_7 343 = 3}$

Math Handbook

of Formulas, Processes and Tricks

(www.mathguy.us)

Geometry

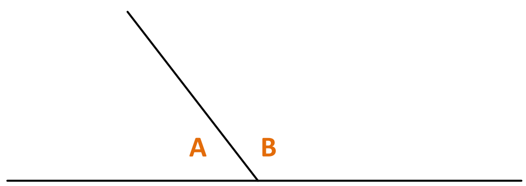


Prepared by: Earl L. Whitney, FSA, MAAA

Version 3.0

April 10, 2017

Geometry
Types of Angles

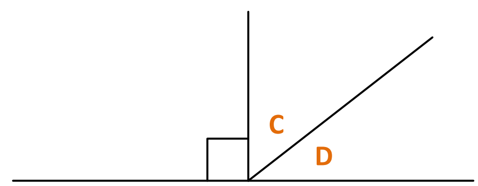


Supplementary Angles

Angles A and B are supplementary.

Angles A and B form a linear pair.

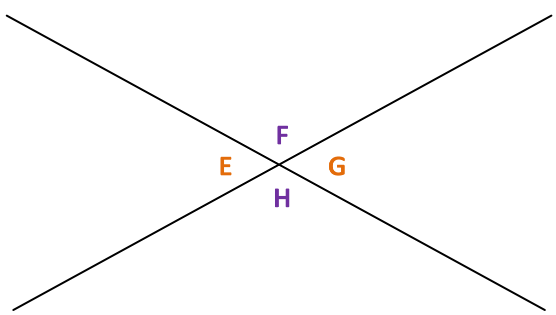
$m\angle A + m\angle B = 180^0$



Complementary Angles

Angles C and D are complementary.

$m\angle C + m\angle D = 90^0$



Vertical Angles

Angles which are opposite each other when two lines cross are vertical angles.

Angles E and G are vertical angles.
Angles F and H are vertical angles.

$m\angle E = m\angle G \quad and \quad m\angle F = m\angle H$

In addition, each angle is supplementary to the two angles adjacent to it. For example:

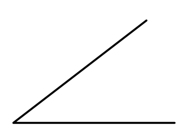
Angle E is supplementary to Angles F and H.

An acute angle is one that is less than 90⁰. In the illustration above, angles E and G are acute angles.

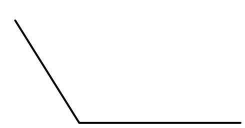
A right angle is one that is exactly 90⁰.

An obtuse angle is one that is greater than 90⁰. In the illustration above, angles F and H are obtuse angles.

A straight angle is one that is exactly 180⁰.



Acute

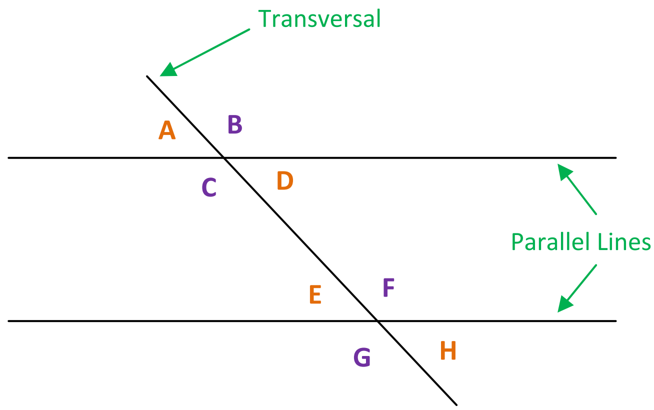


Obtuse

Right

Straight

Geometry
Parallel Lines and Transversals



Alternate: refers to angles that are on opposite sides of the transversal.

Consecutive: refers to angles that are on the same side of the transversal.

Interior: refers to angles that are between the parallel lines.

Exterior: refers to angles that are outside the parallel lines.

Corresponding Angles

Corresponding Angles are angles in the same location relative to the parallel lines and the transversal. For example, the angles on top of the parallel lines and left of the transversal (i.e., top left) are corresponding angles.

Angles **A** and **E (top left)** are **Corresponding Angles**. So are angle pairs **B** and **F (top right)**, **C** and **G (bottom left)**, and **D** and **H (bottom right)**. Corresponding angles are congruent.

Alternate Interior Angles

Angles **D** and **E** are **Alternate Interior Angles**. Angles **C** and **F** are also alternate interior angles. Alternate interior angles are congruent.

Alternate Exterior Angles

Angles **A** and **H** are **Alternate Exterior Angles**. Angles **B** and **G** are also alternate exterior angles. Alternate exterior angles are congruent.

Consecutive Interior Angles

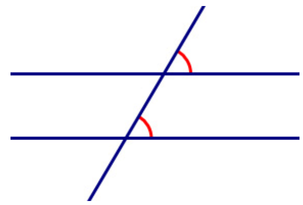
Angles **C** and **E** are **Consecutive Interior Angles**. Angles **D** and **F** are also consecutive interior angles. Consecutive interior angles are supplementary.

*Note that angles **A**, **D**, **E**, and **H** are congruent, and angles **B**, **C**, **F**, and **G** are congruent. In addition, each of the angles in the first group are supplementary to each of the angles in the second group.*

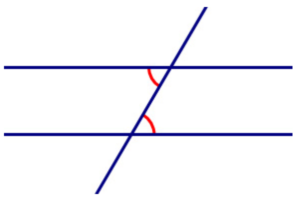
Geometry
Proving Lines are Parallel

The properties of parallel lines cut by a transversal can be used to prove two lines are parallel.

Corresponding Angles



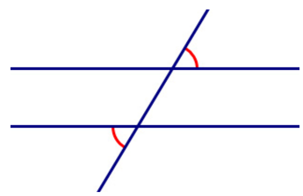
If two lines cut by a transversal have congruent corresponding angles, then the lines are parallel. Note that there are 4 sets of corresponding angles.



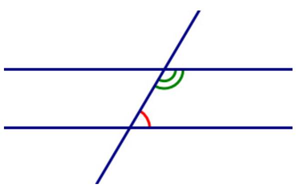
Alternate Interior Angles

If two lines cut by a transversal have congruent alternate interior angles congruent, then the lines are parallel. Note that there are 2 sets of alternate interior angles.

Alternate Exterior Angles



If two lines cut by a transversal have congruent alternate exterior angles, then the lines are parallel. Note that there are 2 sets of alternate exterior angles.



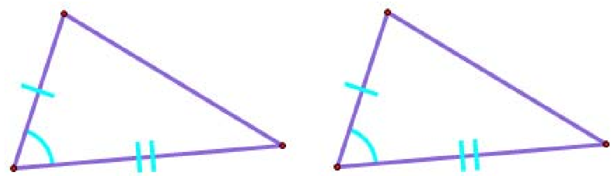
Consecutive Interior Angles

If two lines cut by a transversal have supplementary consecutive interior angles, then the lines are parallel. Note that there are 2 sets of consecutive interior angles.

Geometry
Congruent Triangles

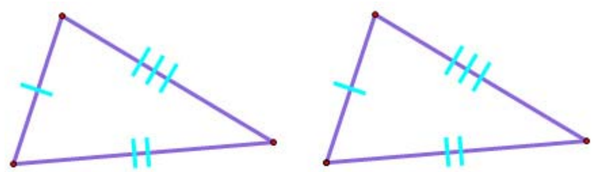
The following theorems present conditions under which triangles are congruent.

Side-Angle-Side (SAS) Congruence



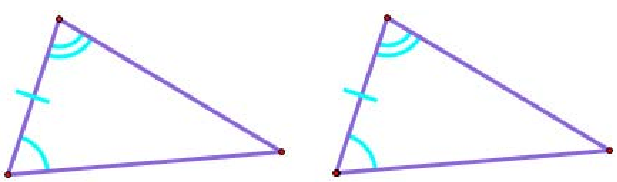
SAS congruence requires the congruence of two sides and the angle between those sides. Note that there is no such thing as SSA congruence; the congruent angle must be between the two congruent sides.

Side-Side-Side (SSS) Congruence



SSS congruence requires the congruence of all three sides. If all of the sides are congruent then all of the angles must be congruent. The converse is not true; there is no such thing as AAA congruence.

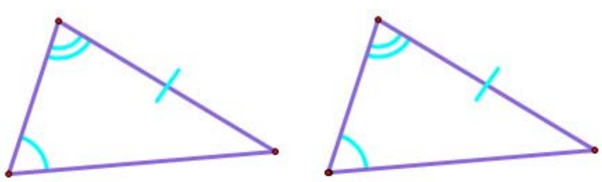
Angle-Side-Angle (ASA) Congruence



ASA congruence requires the congruence of two angles and the side between those angles.

Note: ASA and AAS combine to provide congruence of two triangles whenever any two angles and any one side of the triangles are congruent.

Angle-Angle-Side (AAS) Congruence



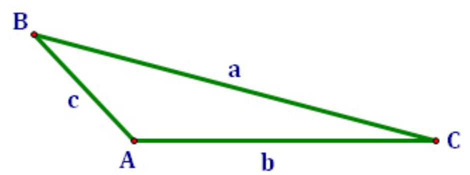
AAS congruence requires the congruence of two angles and a side which is not between those angles.

CPCTC

CPCTC means **"corresponding parts of congruent triangles are congruent."** It is a very powerful tool in geometry proofs and is often used shortly after a step in the proof where a pair of triangles is proved to be congruent.

Geometry
Inequalities in Triangles

Angles and their opposite sides in triangles are related. In fact, this is often reflected in the labeling of angles and sides in triangle illustrations.



Angles and their opposite sides are often labeled with the same letter. An upper case letter is used for the angle and a lower case letter is used for the side.

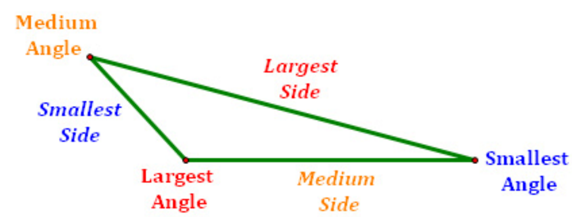
The relationship between angles and their opposite sides translates into the following triangle inequalities:

$$\text{If } \ m\angle C < m\angle B < m\angle A, \ \text{ then } \ c < b < a$$

$$\text{If } \ m\angle C \le m\angle B \le m\angle A, \ \text{ then } \ c \le b \le a$$

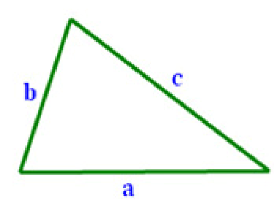
That is, in any triangle,

- The largest side is opposite the largest angle.
- The medium side is opposite the medium angle.
- The smallest side is opposite the smallest angle.



Other Inequalities in Triangles

Triangle Inequality: The sum of the lengths of any two sides of a triangle is greater than the length of the third side. This is a crucial element in deciding whether segments of any 3 lengths can form a triangle.



$$a + b > c \quad \text{and} \quad b + c > a \quad \text{and} \quad c + a > b$$

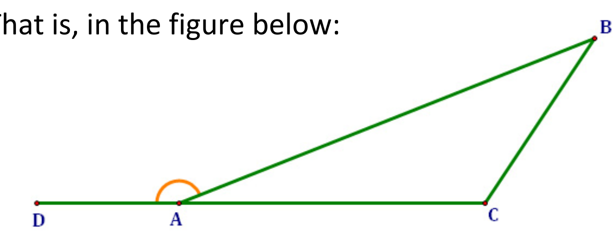
Exterior Angle Inequality: The measure of an external angle is greater than the measure of either of the two non-adjacent interior angles. That is, in the figure below:

$$m\angle DAB > m\angle B \quad \text{and} \quad m\angle DAB > m\angle C$$

Note: the Exterior Angle Inequality is much less relevant than the Exterior Angle Equality.

Exterior Angle Equality: The measure of an external angle is equal to the sum of the measures of the two non-adjacent interior angles. That is, in the figure below:

$$m\angle DAB = m\angle B + m\angle C$$



Geometry
Polygons - Basics

Basic Definitions

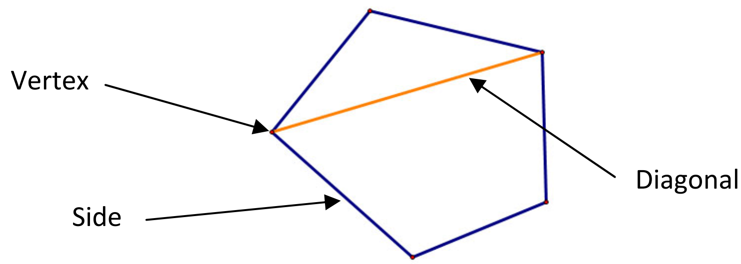
Polygon: a closed path of three or more line segments, where:

- no two sides with a common endpoint are collinear, and
- each segment is connected at its endpoints to exactly two other segments.

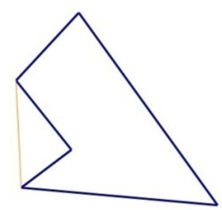
Side: a segment that is connected to other segments (which are also sides) to form a polygon.

Vertex: a point at the intersection of two sides of the polygon. (plural form: **vertices**)

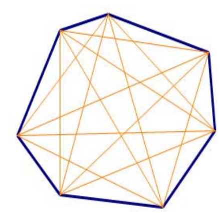
Diagonal: a segment, from one vertex to another, which is not a side.



Concave: A polygon in which it is possible to draw a diagonal "outside" the polygon. (Notice the orange diagonal drawn outside the polygon at right.) Concave polygons actually look like they have a "cave" in them.



Convex: A polygon in which it is not possible to draw a diagonal "outside" the polygon. (Notice that all of the orange diagonals are inside the polygon at right.) Convex polygons appear more "rounded" and do not contain "caves."



Names of Some Common Polygons

Number of Sides	Name of Polygon	Number of Sides	Name of Polygon
3	Triangle	9	Nonagon
4	Quadrilateral	10	Decagon
5	Pentagon	11	Undecagon
6	Hexagon	12	Dodecagon
7	Heptagon	20	Icosagon
8	Octagon	n	n-gon

Names of polygons are generally formed from the Greek language; however, some hybrid forms of Latin and Greek (e.g., undecagon) have crept into common usage.

Geometry
Interior and Exterior Angles of a Polygon

Interior Angles

The sum of the interior angles in an n-sided polygon is:

$$\sum = (n-2) \cdot 180°$$

If the polygon is regular, you can calculate the measure of each interior angle as:

$$\frac{(n-2) \cdot 180°}{n}$$

Notation: The Greek letter **"Σ"** is equivalent to the English letter **"S"** and is math short-hand for a summation (i.e., addition) of things.

Interior Angles		
Sides	**Sum of Interior Angles**	**Each Interior Angle**
3	180^o	60^o
4	360^o	90^o
5	540^o	108^o
6	720^o	120^o
7	900^o	129^o
8	$1,080^o$	135^o
9	$1,260^o$	140^o
10	$1,440^o$	144^o

Exterior Angles

No matter how many sides there are in a polygon, the sum of the exterior angles is:

$$\sum = 360^0$$

If the polygon is regular, you can calculate the measure of each exterior angle as:

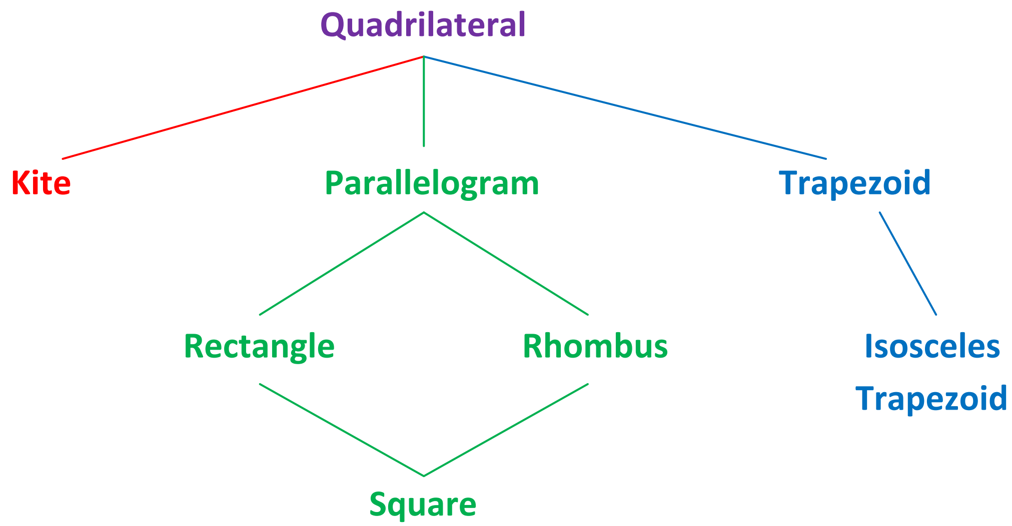
$$\frac{360^0}{n}$$

Exterior Angles		
Sides	**Sum of Exterior Angles**	**Each Exterior Angle**
3	360^o	120^o
4	360^o	90^o
5	360^o	72^o
6	360^o	60^o
7	360^o	51^o
8	360^o	45^o
9	360^o	40^o
10	360^o	36^o

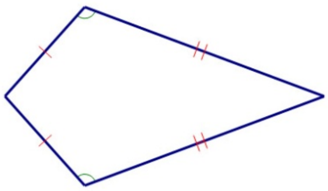
Geometry
Definitions of Quadrilaterals

Name	Definition
Quadrilateral	A polygon with 4 sides.
Kite	A quadrilateral with two consecutive pairs of congruent sides, but with opposite sides not congruent.
Trapezoid	A quadrilateral with exactly one pair of parallel sides.
Isosceles Trapezoid	A trapezoid with congruent legs.
Parallelogram	A quadrilateral with both pairs of opposite sides parallel.
Rectangle	A parallelogram with all angles congruent (i.e., right angles).
Rhombus	A parallelogram with all sides congruent.
Square	A quadrilateral with all sides congruent and all angles congruent.

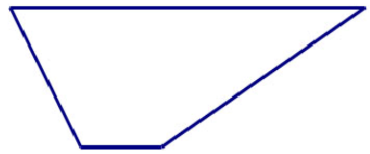
Quadrilateral Tree:



Geometry
Figures of Quadrilaterals

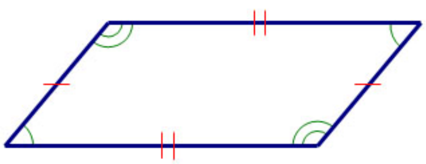


Kite
- 2 consecutive pairs of congruent sides
- 1 pair of congruent opposite angles
- Diagonals perpendicular

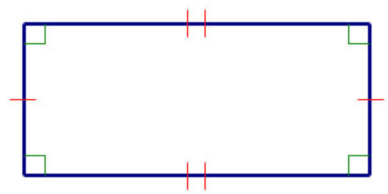


Trapezoid
- 1 pair of parallel sides (called "bases")
- Angles on the same "side" of the bases are supplementary

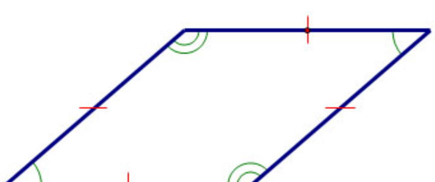
Isosceles Trapezoid
- 1 pair of parallel sides
- Congruent legs
- 2 pair of congruent base angles
- Diagonals congruent



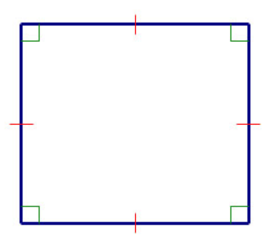
Parallelogram
- Both pairs of opposite sides parallel
- Both pairs of opposite sides congruent
- Both pairs of opposite angles congruent
- Consecutive angles supplementary
- Diagonals bisect each other



Rectangle
- Parallelogram with all angles congruent (i.e., right angles)
- Diagonals congruent



Rhombus
- Parallelogram with all sides congruent
- Diagonals perpendicular
- Each diagonal bisects a pair of opposite angles



Square
- Both a Rhombus and a Rectangle
- All angles congruent (i.e., right angles)
- All sides congruent

Geometry
Introduction to Transformation

A **Transformation** is a mapping of the pre-image of a geometric figure onto an image that retains key characteristics of the pre-image.

Definitions

The **Pre-Image** is the geometric figure before it has been transformed.

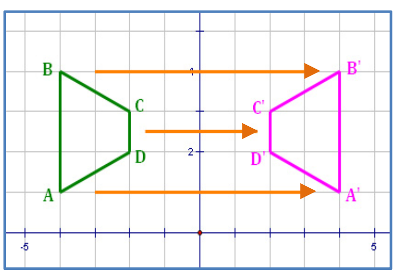
The **Image** is the geometric figure after it has been transformed.

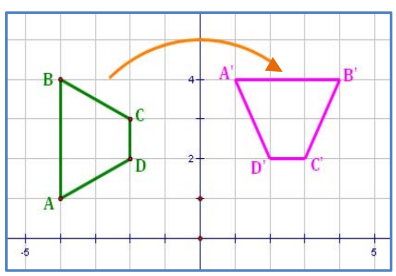
A **mapping** is an association between objects. Transformations are types of mappings. In the figures below, we say *ABCD is mapped onto A'B'C'D'*, or $ABCD \longrightarrow A'B'C'D'$. The order of the vertices is critical to a properly named mapping.

An **Isometry** is a one-to-one mapping that preserves lengths. Transformations that are isometries (i.e., preserve length) are called **rigid transformations**.

Isometric Transformations

Reflection is flipping a figure across a line called a "mirror." The figure retains its size and shape, but appears "backwards" after the reflection.	**Rotation** is turning a figure around a point. Rotated figures retain their size and shape, but not their orientation.	**Translation** is sliding a figure in the plane so that it changes location but retains its shape, size and orientation.





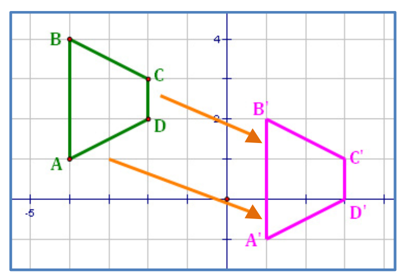


Table of Characteristics of Isometric Transformations

Transformation	Reflection	Rotation	Translation
Isometry (Retains Lengths)?	Yes	Yes	Yes
Retains Angles?	Yes	Yes	Yes
Retains Orientation to Axes?	No	No	Yes

Geometry
Introduction to Transformation (cont'd)

Transformation of a Point

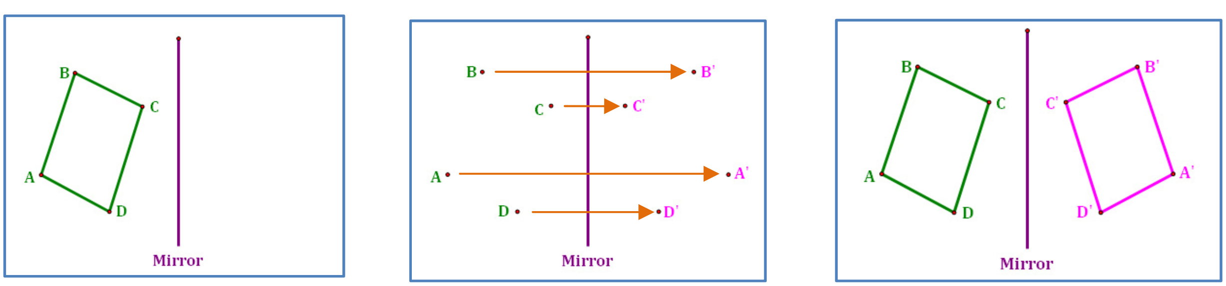
A point is the easiest object to transform. Simply reflect, rotate or translate it following the rules for the transformation selected. By transforming key points first, any transformation becomes much easier.

Transformation of a Geometric Figure

To transform any geometric figure, it is only necessary to transform the items that define the figure, and then re-form it. For example:

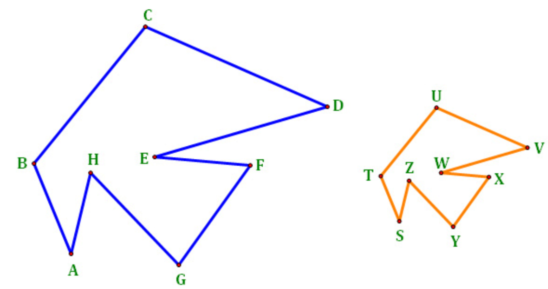
- To transform a **line segment**, transform its two endpoints, and then connect the resulting images with a line segment.
- To transform a **ray**, transform the initial point and any other point on the ray, and then construct a ray using the resulting images.
- To transform a **line**, transform any two points on the line, and then fit a line through the resulting images.
- To transform a **polygon**, transform each of its vertices, and then connect the resulting images with line segments.
- To transform a **circle**, transform its center and, if necessary, its radius. From the resulting images, construct the image circle.
- To transform **other conic sections (parabolas, ellipses and hyperbolas)**, transform the foci, vertices and/or directrix. From the resulting images, construct the image conic section.

Example: Reflect Quadrilateral ABCD



Geometry
Similar Polygons

In similar polygons,



- Corresponding angles are congruent, and
- Corresponding sides are proportional.

Both of these conditions are necessary for two polygons to be similar. Conversely, when two polygons are similar, all of the corresponding angles are congruent and all of the sides are proportional.

Naming Similar Polygons

Similar polygons should be named such that corresponding angles are in the same location in the name, and the order of the points in the name should "follow the polygon around."

Example: The polygons above could be shown similar with the following names:

$$ABCDEFGHI \sim STUVWXYZ$$

It would also be acceptable to show the similarity as:

$$DEFGHIABC \sim VWXYZSTU$$

Any names that preserve the order of the points and keeps corresponding angles in corresponding locations in the names would be acceptable.

Proportions

One common problem relating to similar polygons is to present three side lengths, where two of the sides correspond, and to ask for the length of the side corresponding to the third length.

Example: In the above similar polygons, if $BC = 20, EF = 12, and\ WX = 6, what\ is\ TU?$

This problem is solvable with proportions. To do so properly, it is important to relate corresponding items in the proportion:

$$\frac{BC}{TU} = \frac{EF}{WX} \quad \longrightarrow \quad \frac{20}{TU} = \frac{12}{6} \quad \longrightarrow \quad TU = 10$$

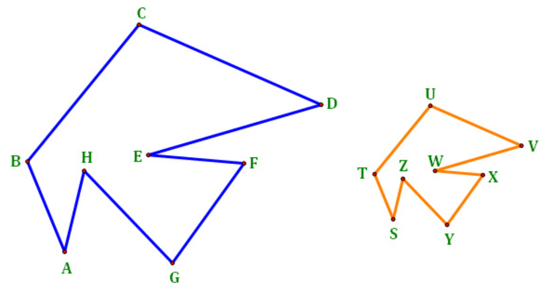
Notice that the left polygon is represented on the top of both proportions and that the left-most segments of the two polygons are in the left fraction.

Geometry
Scale Factors of Similar Polygons

From the similar polygons below, the following is known about the lengths of the sides:

$$\frac{AB}{ST} = \frac{BC}{TU} = \frac{CD}{UV} = \frac{DE}{VW} = \frac{EF}{WX} = \frac{FG}{XY} = \frac{GH}{YZ} = \frac{HA}{ZA} = k$$

That is, the ratios of corresponding sides in the two polygons are the same and they equal some constant k, called the **scale factor** of the two polygons. The value of k, then, is all you need to know to relate corresponding sides in the two polygons.



Finding the Missing Length

Any time the student is asked to find the missing length in similar polygons:

- Look for two corresponding sides for which the values are known.
- Calculate the value of k.
- Use the value of k to solve for the missing length.

k is a measure of the relative size of the two polygons. Using this knowledge, it is possible to put into words an easily understandable relationship between the polygons.

- Let Polygon 1 be the one whose sides are in the numerators of the fractions.
- Let Polygon 2 be the one whose sides are in the denominators of the fractions.
- Then, it can be said that **Polygon 1 is k times the size of the Polygon 2**.

Example: In the above similar polygons, if $BC = 20, EF = 12, and\ WX = 6, what\ is\ TU?$

Seeing that EF and WX relate, calculate:

$$\frac{EF}{WX} = \frac{12}{6} = 2 = k$$

Then solve for TU based on the value of k:

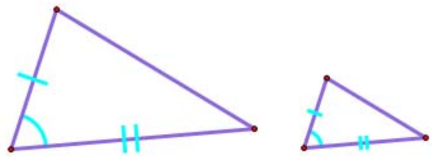
$$\frac{BC}{TU} = k \quad \rightarrow \quad \frac{20}{TU} = 2 \quad \rightarrow \quad TU = 10$$

Also, since $k = 2$, the length of every side in the blue polygon is double the length of its corresponding side in the orange polygon.

Geometry
Similar Triangles

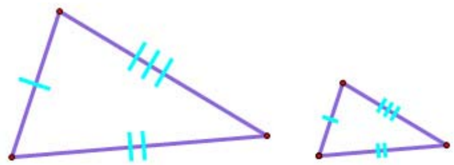
The following theorems present conditions under which triangles are similar.

Side-Angle-Side (SAS) Similarity



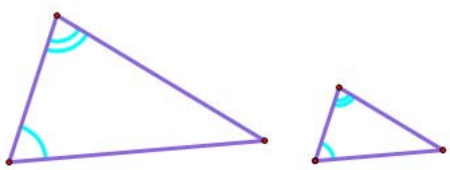
SAS similarity requires the proportionality of two sides and the congruence of the angle between those sides. Note that there is no such thing as SSA similarity; the congruent angle must be between the two proportional sides.

Side-Side-Side (SSS) Similarity



SSS similarity requires the proportionality of all three sides. If all of the sides are proportional, then all of the angles must be congruent.

Angle--Angle (AA) Similarity

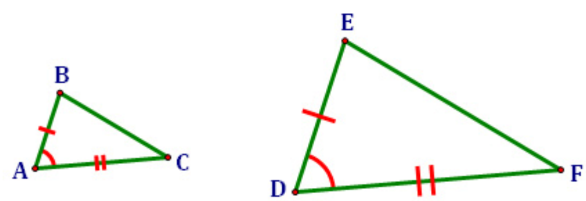


AA similarity requires the congruence of two angles and the side between those angles.

Similar Triangle Parts

In similar triangles,

- Corresponding sides are proportional.
- Corresponding angles are congruent.



Establishing the proper names for similar triangles is crucial to line up corresponding vertices. In the picture above, we can say:

$$\Delta ABC \sim \Delta DEF \quad \text{or} \quad \Delta BCA \sim \Delta EFD \quad \text{or} \quad \Delta CAB \sim \Delta FDE \quad \text{or}$$
$$\Delta ACB \sim \Delta DFE \quad \text{or} \quad \Delta BAC \sim \Delta EDF \quad \text{or} \quad \Delta CBA \sim \Delta FED$$

All of these are correct because they match corresponding parts in the naming. Each of these similarities implies the following relationships between parts of the two triangles:

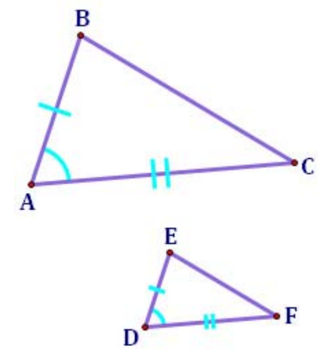
$$\angle A \cong \angle D \quad \text{and} \quad \angle B \cong \angle E \quad \text{and} \quad \angle C \cong \angle F$$

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

Geometry
Proportion Tables for Similar Triangles

Setting Up a Table of Proportions

It is often useful to set up a table to identify the proper proportions in a similarity. Consider the figure to the right. The table might look something like this:



Triangle	Left Side	Right Side	Bottom Side
Top Δ	AB	BC	CA
Bottom Δ	DE	EF	FD

The purpose of a table like this is to organize the information you have about the similar triangles so that you can readily develop the proportions you need.

Developing the Proportions

To develop proportions from the table:

- Extract the columns needed from the table:

AB	BC
DE	EF

- Eliminate the table lines.
- Replace the horizontal lines with "division lines."
- Put an equal sign between the two resulting fractions:

$$\frac{AB}{DE} = \frac{BC}{EF}$$

Also from the above table,

$$\frac{AB}{DE} = \frac{CA}{FD}$$

$$\frac{BC}{EF} = \frac{CA}{FD}$$

Solving for the unknown length of a side:

You can extract any two columns you like from the table. Usually, you will have information on lengths of three of the sides and will be asked to calculate a fourth.

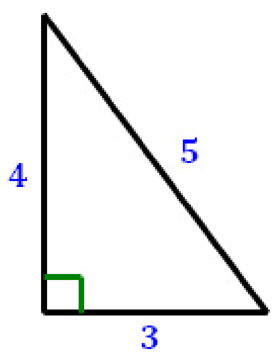
Look in the table for the columns that contain the 4 sides in question, and then set up your proportion. Substitute known values into the proportion, and solve for the remaining variable.

Geometry
Pythagorean Triples

Pythagorean Theorem: $a^2 + b^2 = c^2$

Pythagorean triples are sets of 3 positive integers that meet the requirements of the Pythagorean Theorem. Because these sets of integers provide "pretty" solutions to geometry problems, they are a favorite of geometry books and teachers. Knowing what triples exist can help the student quickly identify solutions to problems that might otherwise take considerable time to solve.

3-4-5 Triangle Family

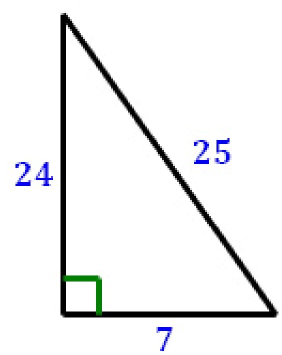


Sample Triples

3-4-5
6-8-10
9-12-15
12-16-20
30-40-50

$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$

7-24-25 Triangle Family

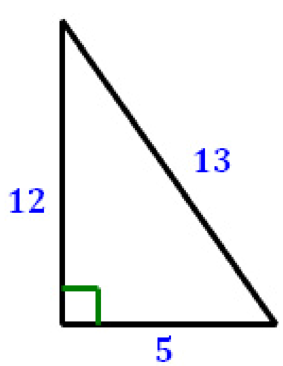


Sample Triples

7-24-25
14-48-50
21-72-75
. . .
70-240-250

$$7^2 + 24^2 = 25^2$$
$$49 + 576 = 625$$

5-12-13 Triangle Family

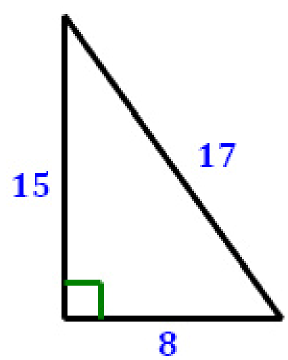


Sample Triples

5-12-13
10-24-26
15-36-39
. . .
50-120-130

$$5^2 + 12^2 = 13^2$$
$$25 + 144 = 169$$

8-15-17 Triangle Family



Sample Triples

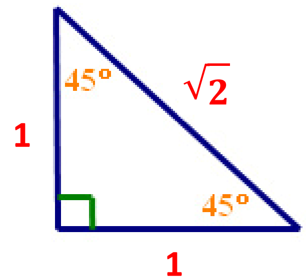
8-15-17
16-30-34
24-45-51
. . .
80-150-170

$$8^2 + 15^2 = 17^2$$
$$64 + 225 = 289$$

Geometry
Special Triangles

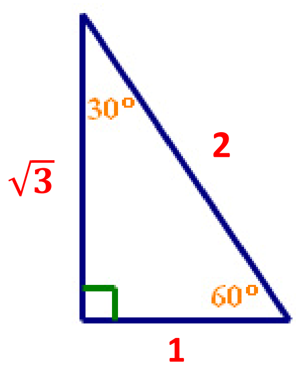
The relationship among the lengths of the sides of a triangle is dependent on the measures of the angles in the triangle. For a right triangle (i.e., one that contains a 90^o angle), two special cases are of particular interest. These are shown below:

45^o-45^o-90^o Triangle



In a **45^o-45^o-90^o triangle**, the congruence of two angles guarantees the congruence of the two legs of the triangle. The proportions of the three sides are: $\mathbf{1 : 1 : \sqrt{2}}$. That is, the two legs have the same length and the hypotenuse is $\sqrt{2}$ times as long as either leg.

30^o-60^o-90^o Triangle



In a **30^o-60^o-90^o triangle**, the proportions of the three sides are: $\mathbf{1 : \sqrt{3} : 2}$. That is, the long leg is $\sqrt{3}$ times as long as the short leg, and the hypotenuse is $\mathbf{2}$ times as long as the short leg.

In a right triangle, we need to know the lengths of two sides to determine the length of the third. The power of the relationships in the special triangles lies in the fact that we need only know the length of one side of the triangle to determine the lengths of the other two sides.

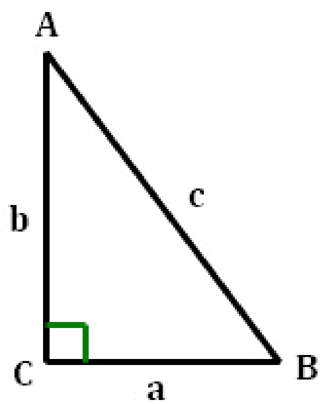
Example Side Lengths

45^o-45^o-90^o Triangle	
$1 : 1 : \sqrt{2}$	$2 : 2 : 2\sqrt{2}$
$\sqrt{2} : \sqrt{2} : 2$	$\sqrt{3} : \sqrt{3} : \sqrt{6}$
$3\sqrt{2} : 3\sqrt{2} : 6$	$25 : 25 : 25\sqrt{2}$

30^o-60^o-90^o Triangle	
$1 : \sqrt{3} : 2$	$2 : 2\sqrt{3} : 4$
$\sqrt{2} : \sqrt{6} : 2\sqrt{2}$	$\sqrt{3} : 3 : 2\sqrt{3}$
$3\sqrt{2} : 3\sqrt{6} : 6\sqrt{2}$	$25 : 25\sqrt{3} : 50$

Geometry
Trig Functions and Special Angles

Trigonometric Functions



SOH-CAH-TOA

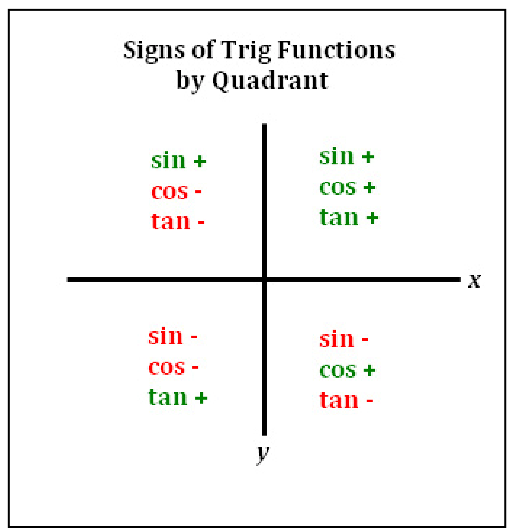
$$\sin = \frac{opposite}{hypoteneuse} \qquad \sin A = \frac{a}{c} \qquad \sin B = \frac{b}{c}$$

$$\cos = \frac{adjacent}{hypoteneuse} \qquad \cos A = \frac{b}{c} \qquad \cos B = \frac{a}{c}$$

$$\tan = \frac{opposite}{adjacent} \qquad \tan A = \frac{a}{b} \qquad \tan B = \frac{b}{a}$$

Special Angles

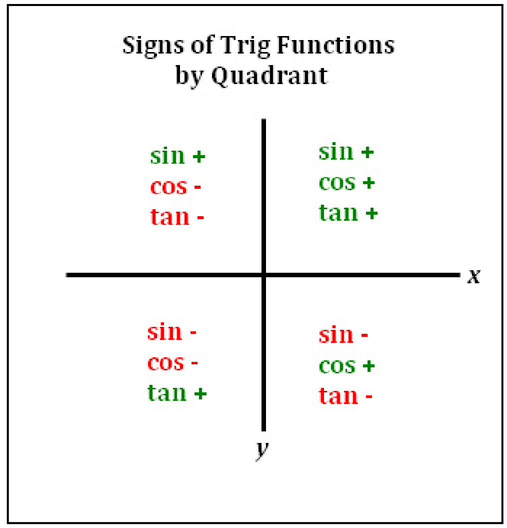
Trig Functions of Special Angles				
Radians	**Degrees**	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0º	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{\sqrt{4}} = 0$
$\pi/6$	30º	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\pi/4$	45º	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{\sqrt{2}} = 1$
$\pi/3$	60º	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$
$\pi/2$	90º	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{2} = 0$	*undefined*



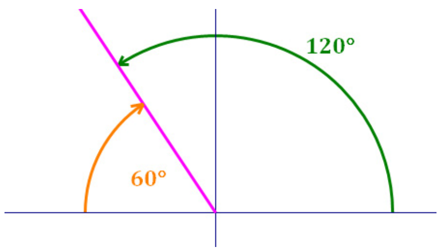
Geometry
Trigonometric Function Values in Quadrants II, III, and IV

In quadrants other than Quadrant I, trigonometric values for angles are calculated in the following manner:

- Draw the angle $\boldsymbol{\theta}$ on the Cartesian Plane.
- Calculate the measure of the angle from the x-axis to $\boldsymbol{\theta}$.
- Find the value of the trigonometric function of the angle in the previous step.
- Assign a "+" or "–" sign to the trigonometric value based on the function used and the quadrant $\boldsymbol{\theta}$ is in.



Examples:



***θ* in Quadrant II** – Calculate: $(180^0 - m\angle\theta)$

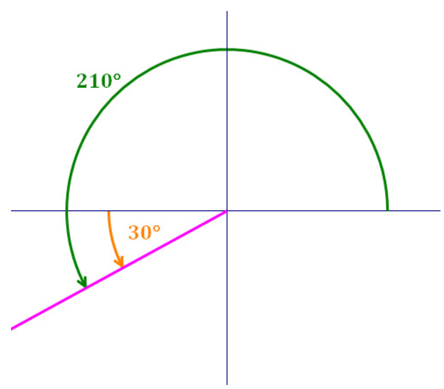
For $\theta = 120^0$, base your work on $180° - 120° = 60°$

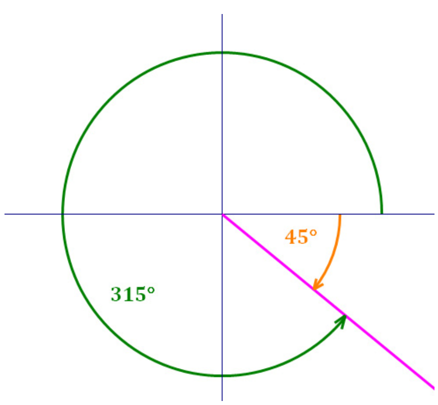
$\sin 60° = \frac{\sqrt{3}}{2}$, so: $\mathbf{\sin 120° = \frac{\sqrt{3}}{2}}$

***θ* in Quadrant III** – Calculate: $(m\angle\theta - 180^0)$

For $\theta = 210^0$, base your work on $210° - 180° = 30°$

$\cos 30° = \frac{\sqrt{3}}{2}$, so: $\mathbf{\cos 210° = -\frac{\sqrt{3}}{2}}$



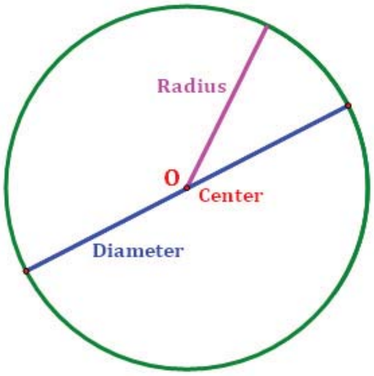


***θ* in Quadrant IV** – Calculate: $(360^0 - m\angle\theta)$

For $\theta = 315^0$, base your work on $360° - 315° = 45°$

$\tan 45° = 1$, so: $\mathbf{\tan 315° = -1}$

Geometry
Parts of Circles



Center – the middle of the circle. All points on the circle are the same distance from the center.

Radius – a line segment with one endpoint at the center and the other endpoint on the circle. The term "radius" is also used to refer to the distance from the center to the points on the circle.

Diameter – a line segment with endpoints on the circle that passes through the center.

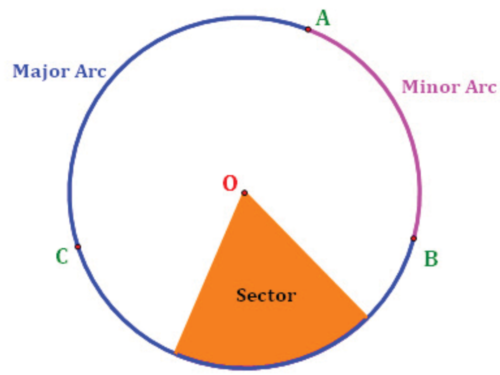
Arc – a path along a circle.

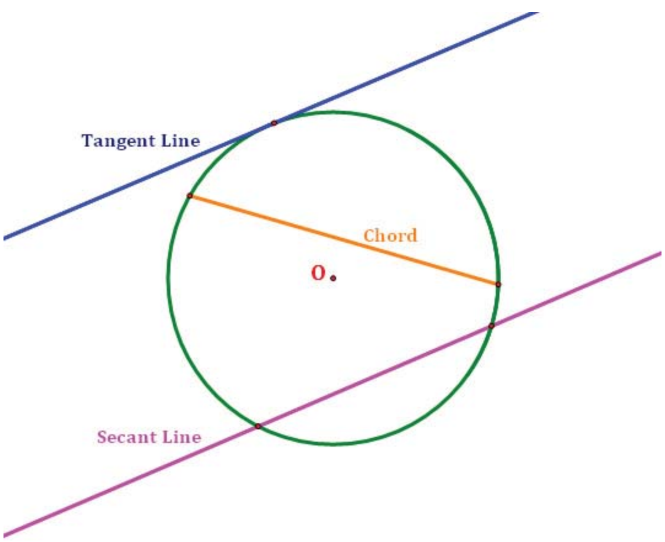
Minor Arc – a path along the circle that is less than 180°.

Major Arc – a path along the circle that is greater than 180°.

Semicircle – a path along a circle that equals 180°.

Sector – a region inside a circle that is bounded by two radii and an arc.





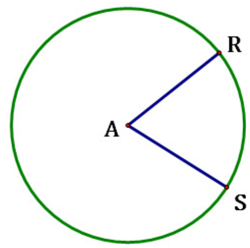
Secant Line – a line that intersects the circle in exactly two points.

Tangent Line– a line that intersects the circle in exactly one point.

Chord – a line segment with endpoints on the circle that does not pass through the center.

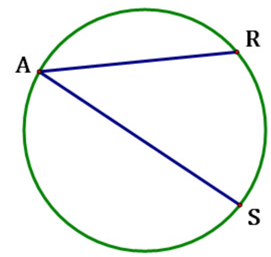
Geometry
Angles and Circles

Central Angle



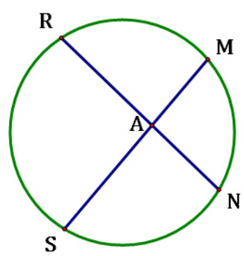
$$m\angle A = m\,\widehat{RS}$$

Inscribed Angle



$$m\angle A = \frac{1}{2}\,m\,\widehat{RS}$$

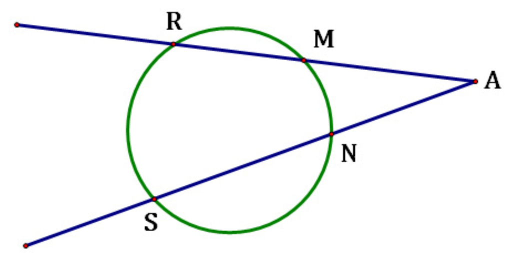
Vertex inside the circle



$$m\angle A = \frac{1}{2}\left(m\,\widehat{RS} + m\,\widehat{MN}\right)$$

$$RA \cdot AN = SA \cdot AM$$

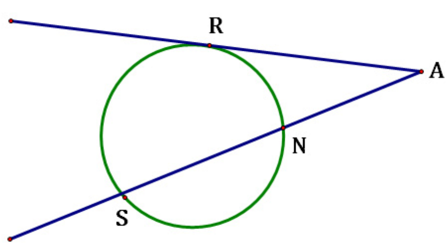
Vertex outside the circle



$$m\angle A = \frac{1}{2}\left(m\,\widehat{RS} - m\,\widehat{MN}\right)$$

$$AM \cdot AR = AN \cdot AS$$

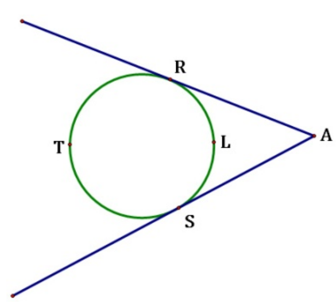
Tangent on one side



$$m\angle A = \frac{1}{2}\left(m\,\widehat{RS} - m\,\widehat{RN}\right)$$

$$AR^2 = AN \cdot AS$$

Tangents on two sides



$$m\angle A = \frac{1}{2}\left(m\,\widehat{RTS} - m\,\widehat{RLS}\right)$$

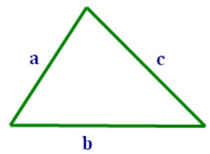
$$AR = AS$$

Geometry
Perimeter and Area of a Triangle

Perimeter of a Triangle

The perimeter of a triangle is simply the sum of the measures of the three sides of the triangle.

$$P = a + b + c$$

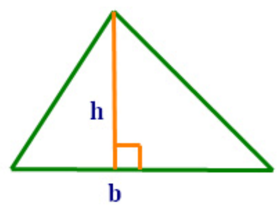


Area of a Triangle

There are two formulas for the area of a triangle, depending on what information about the triangle is available.

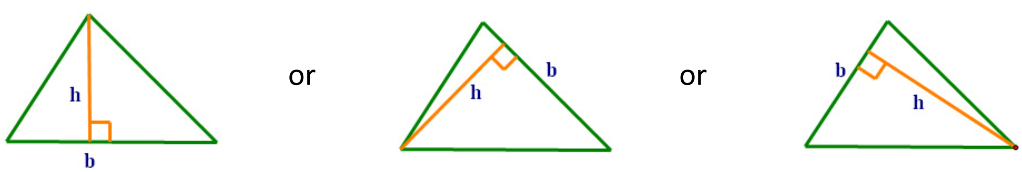
Formula 1: The formula most familiar to the student can be used when the base and height of the triangle are either known or can be determined.

$$A = \frac{1}{2}bh$$

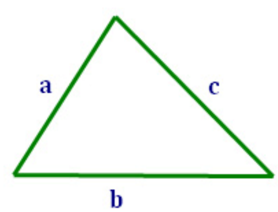


where, b is the length of the base of the triangle.
h is the height of the triangle.

Note: The base can be any side of the triangle. The height is the measure of the altitude of whichever side is selected as the base. So, you can use:



Formula 2: Heron's formula for the area of a triangle can be used when the lengths of all of the sides are known. Sometimes this formula, though less appealing, can be very useful.



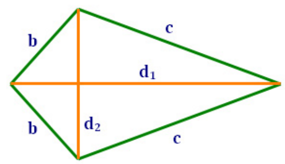
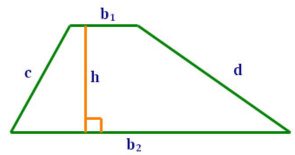
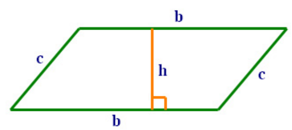
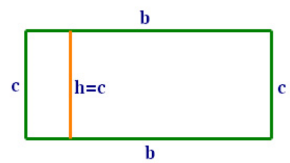
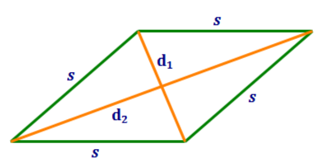
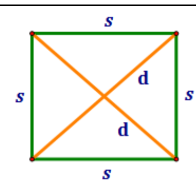
$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where, $s = \frac{1}{2}P = \frac{1}{2}(a+b+c)$. **Note:** s is sometimes called the semi-perimeter of the triangle.

a, b, c are the lengths of the sides of the triangle.

Geometry

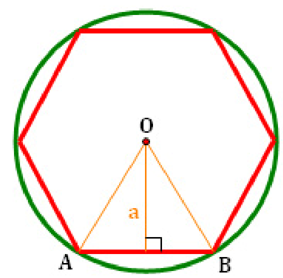
Perimeter and Area of Quadrilaterals

Name	Illustration	Perimeter	Area
Kite		$P = 2b + 2c$	$A = \frac{1}{2}(d_1 d_2)$
Trapezoid		$P = b_1 + b_2 + c + d$	$A = \frac{1}{2}(b_1 + b_2)h$
Parallelogram		$P = 2b + 2c$	$A = bh$
Rectangle		$P = 2b + 2c$	$A = bh$
Rhombus		$P = 4s$	$A = bh = \frac{1}{2}(d_1 d_2)$
Square		$P = 4s$	$A = s^2 = \frac{1}{2}(d^2)$

Geometry
Perimeter and Area of Regular Polygons

Definitions – Regular Polygons

- The **center** of a polygon is the center of its circumscribed circle. Point $\boldsymbol{O}$ is the center of the hexagon at right.
- The **radius** of the polygon is the radius of its circumscribed circle. $\overline{\boldsymbol{OA}}$ and $\overline{\boldsymbol{OB}}$ are both radii of the hexagon at right.
- The **apothem** of a polygon is the distance from the center to the midpoint of any of its sides. **a** is the apothem of the hexagon at right.



- The **central angle** of a polygon is an angle whose vertex is the center of the circle and whose sides pass through consecutive vertices of the polygon. In the figure above, $\angle\boldsymbol{AOB}$ is a central angle of the hexagon.

Area of a Regular Polygon

$$A = \frac{1}{2}aP$$

where, a is the apothem of the polygon

P is the perimeter of the polygon

Perimeter and Area of Similar Figures

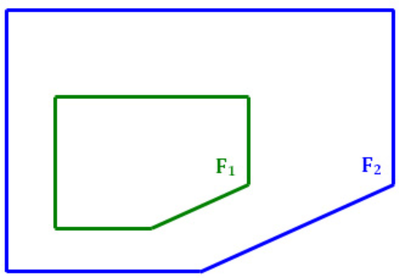
Let $\mathbf{k}$ be the scale factor relating two similar geometric figures $\mathbf{F_1}$ and $\mathbf{F_2}$ such that $\mathbf{F_2} = \mathbf{k} \cdot \mathbf{F_1}$.

Then,

$$\frac{\text{Perimeter of } F_2}{\text{Perimeter of } F_1} = k$$

and

$$\frac{\text{Area of } F_2}{\text{Area of } F_1} = k^2$$



Geometry
Circle Lengths and Areas

Circumference and Area

$\boldsymbol{C = 2\pi \cdot r}$ is the **circumference** (i.e., the perimeter) of the circle.

$\boldsymbol{A = \pi r^2}$ is the **area** of the circle.

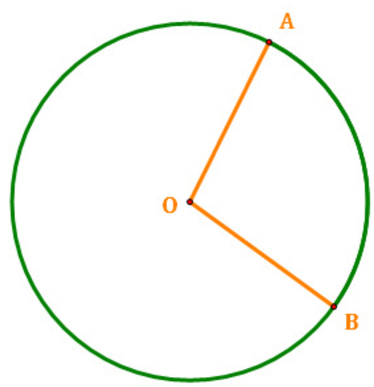
where: r is the radius of the circle.

Length of an Arc on a Circle

A common problem in the geometry of circles is to measure the length of an arc on a circle.

Definition: An arc is a segment along the circumference of a circle.

$$\boldsymbol{arc\ length = \frac{m\widehat{AB}}{360} \cdot C}$$



where: $m\angle\widehat{AB}$ is the measure (in degrees) of the arc. Note that this is also the measure of the central angle $\angle AOB$.

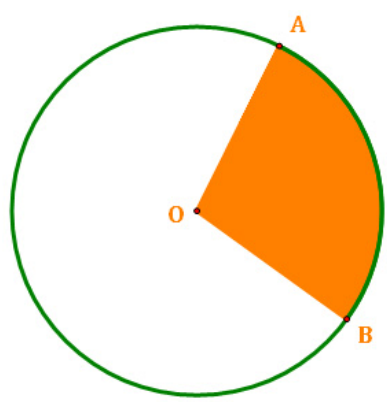
C is the circumference of the circle.

Area of a Sector of a Circle

Another common problem in the geometry of circles is to measure the area of a sector a circle.

Definition: A sector is a region in a circle that is bounded by two radii and an arc of the circle.

$$\boldsymbol{sector\ area = \frac{m\widehat{AB}}{360} \cdot A}$$



where: $m\angle\widehat{AB}$ is the measure (in degrees) of the arc. Note that this is also the measure of the central angle $\angle AOB$.

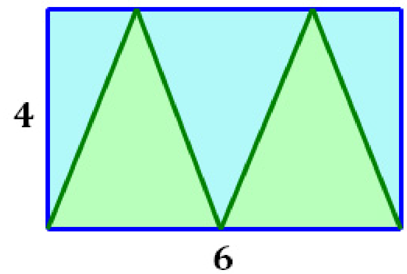
A is the area of the circle.

Geometry
Area of Composite Figures

To calculate the area of a figure that is a composite of shapes, consider each shape separately.

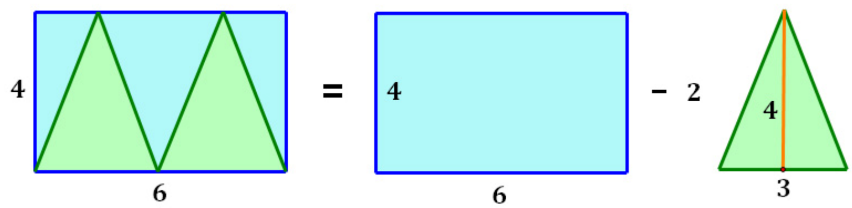
Example 1:

Calculate the area of the blue region in the figure to the right.



To solve this:

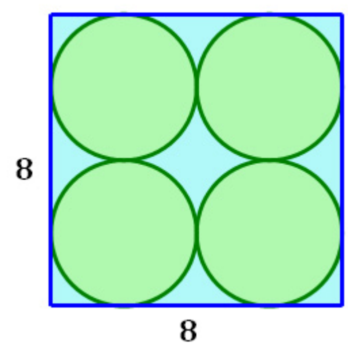
- Recognize that the figure is the composite of a rectangle and two triangles.
- Disassemble the composite figure into its components.
- Calculate the area of the components.
- Subtract to get the area of the composite figure.



$$\textit{Area of Region} = (4 \cdot 6) - 2\left(\frac{1}{2} \cdot 4 \cdot 3\right) = 24 - 12 = 12$$

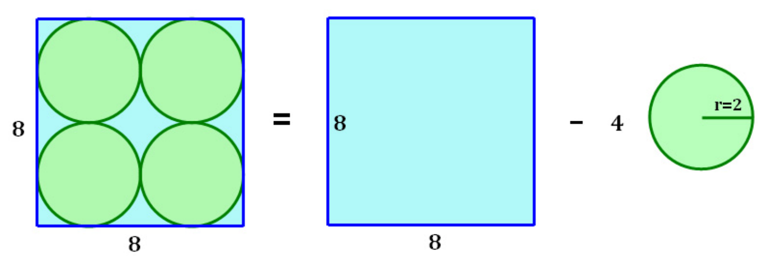
Example 2:

Calculate the area of the blue region in the figure to the right.



To solve this:

- Recognize that the figure is the composite of a square and a circle.
- Disassemble the composite figure into its components.
- Calculate the area of the components.
- Subtract to get the area of the composite figure.



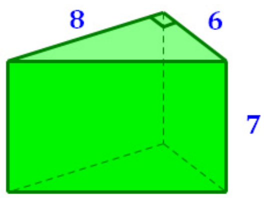
$$\textit{Area of Region} = 8^2 - 4(\pi \cdot 2^2) = 64 - 16\pi \sim 13.73$$

Geometry
Surface Area by Decomposition

Sometimes the student is asked to calculate the surface are of a prism that does not quite fit into one of the categories for which an easy formula exists. In this case, the answer may be to decompose the prism into its component shapes, and then calculate the areas of the components. Note: this process also works with cylinders and pyramids.

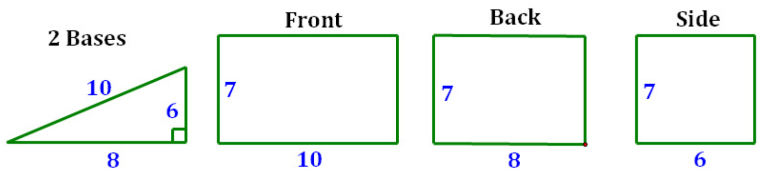
Decomposition of a Prism

To calculate the surface area of a prism, decompose it and look at each of the prism's faces individually.



Example: Calculate the surface area of the triangular prism at right.

To do this, first notice that we need the value of the hypotenuse of the base. Use the Pythagorean Theorem or Pythagorean Triples to determine the missing value is ***10***. Then, decompose the figure into its various faces:



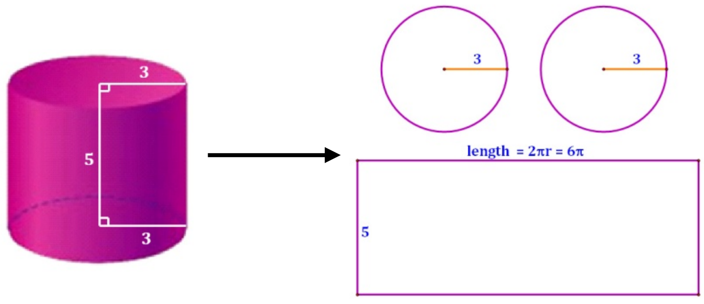
The surface area, then, is calculated as:

$$SA = (2\ Bases) + (Front) + (Back) + (Side)$$

$$SA = 2 \cdot \left(\frac{1}{2} \cdot 6 \cdot 8\right) + (10 \cdot 7) + (8 \cdot 7) + (6 \cdot 7) = 216$$

Decomposition of a Cylinder

The cylinder at right is decomposed into two circles (the bases) and a rectangle (the lateral face).

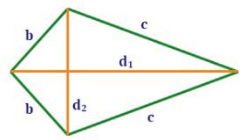
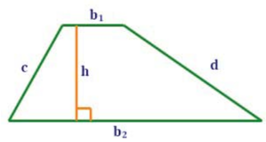
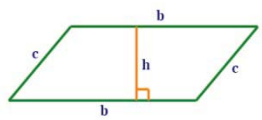
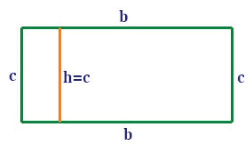
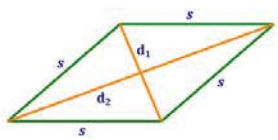
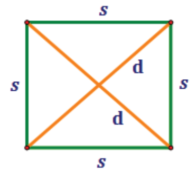
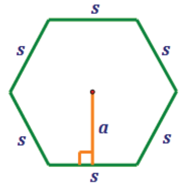
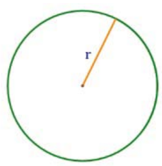
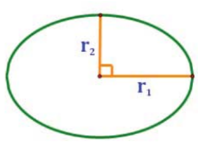


The surface area, then, is calculated as:

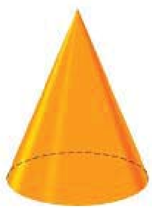
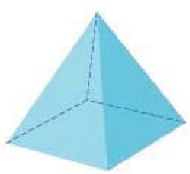
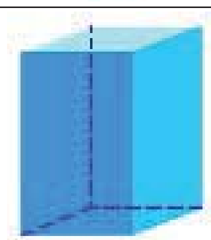
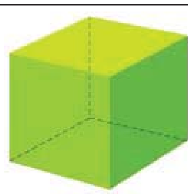
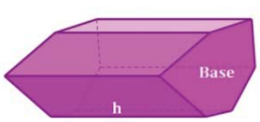
$$SA = (2\ tops) + (lateral\ face)$$

$$SA = 2 \cdot (\pi \cdot 3^2) + (6\pi \cdot 5) = 48\pi \sim 150.80$$

Geometry
Summary of Perimeter and Area Formulas – 2D Shapes

Shape	Figure	Perimeter	Area
Kite		$P = 2b + 2c$ $b, c = sides$	$A = \frac{1}{2}(d_1 d_2)$ $d_1, d_2 = diagonals$
Trapezoid		$P = b_1 + b_2 + c + d$ $b_1, b_2 = bases$ $c, d = sides$	$A = \frac{1}{2}(b_1 + b_2)h$ $b_1, b_2 = \text{bases}$ $h = \text{height}$
Parallelogram		$P = 2b + 2c$ $b, c = sides$	$A = bh$ $b = base$ $h = height$
Rectangle		$P = 2b + 2c$ $b, c = sides$	$A = bh$ $b = base$ $h = height$
Rhombus		$P = 4s$ $s = side$	$A = bh = \frac{1}{2}(d_1 d_2)$ $d_1, d_2 = diagonals$
Square		$P = 4s$ $s = side$	$A = s^2 = \frac{1}{2}(d_1 d_2)$ $d_1, d_2 = diagonals$
Regular Polygon		$P = ns$ $n = number\ of\ sides$ $s = side$	$A = \frac{1}{2}\ a \cdot P$ $a = apothem$ $P = perimeter$
Circle		$C = 2\pi r = \pi d$ $r = radius$ $d = diameter$	$A = \pi r^2$ $r = radius$
Ellipse		$P \approx 2\pi\sqrt{\frac{1}{2}({r_1}^2 + {r_2}^2)}$ $r_1 = major\ axis\ radius$ $r_2 = minor\ axis\ radius$	$A = \pi r_1 r_2$ $r_1 = major\ axis\ radius$ $r_2 = minor\ axis\ radius$

Geometry
Summary of Surface Area and Volume Formulas – 3D Shapes

Shape	Figure	Surface Area	Volume
Sphere		$SA = 4\pi r^2$ $r = radius$	$V = \frac{4}{3}\pi r^3$ $r = radius$
Right Cylinder		$SA = 2\pi rh + 2\pi r^2$ $h = height$ $r = radius\ of\ base$	$V = \pi r^2 h$ $h = height$ $r = radius\ of\ base$
Cone		$SA = \pi rl + \pi r^2$ $l = slant\ height$ $r = radius\ of\ base$	$V = \frac{1}{3}\pi r^2 h$ $h = height$ $r = radius\ of\ base$
Square Pyramid		$SA = 2sl + s^2$ $s = base\ side\ length$ $l = slant\ height$	$V = \frac{1}{3}s^2 h$ $s = base\ side\ length$ $h = height$
Rectangular Prism		$SA = 2 \cdot (lw + lh + wh)$ $l = length$ $w = width$ $h = height$	$V = lwh$ $l = length$ $w = width$ $h = height$
Cube		$SA = 6s^2$ $s = side\ length\ (all\ sides)$	$V = s^3$ $s = side\ length\ (all\ sides)$
General Right Prism		$SA = Ph + 2B$ $P = Perimeter\ of\ Base$ $h = height\ (or\ length)$ $B = area\ of\ Base$	$V = Bh$ $B = area\ of\ Base$ $h = height$