PARTIAL FRACTION DECOMPOSITIONS

We shall examine **three** methods of **speeding up** partial fraction decompositions. Before continuing, however, the reader should review the basic method of partial fraction decomposition in any standard college algebra book.

The first method is called **Heaviside's cover-up technique** and applies to linear factors in the denominator of the fraction. It will produce the constant numerator for the↓highest power↑ of the linear factor. For example, in the decomposition

$$\frac{x^2-3x-5}{(x-1)^3(x+2)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1} + \frac{D}{x+2} \qquad \text{(I)}$$

the values for **A** and for **D** can be obtained by this technique. The values for B and C *cannot*. To obtain B and C one must clear fractions and use the standard technique. But **the work is reduced** since two of the four unknowns are already calculated. This reduces the number of equations required to two instead of four.

To evaluate A we set the factor **x-1=0** and solve for x; ie. **x=1**. Then "**cover up**" (:::) the (x-1) factor and **evaluate the rest** of the expression at x=1 (the value that makes x-1=0).

$$\frac{x^2-3x-5}{(:::)\ (x+2)} = \frac{(1)^2-3(1)-5}{(1+2)} = \frac{-7}{3} = -7/3$$

This value, -7/3, is the value for A, the constant over the **highest power** of the factor (x-1). In a similar fashion, to obtain **D** we solve **x+2=0** for the value **x=-2** used to evaluate the expression after covering up the (x+2) factor:

$$\frac{x^2-3x-5}{(x-1)^3(:::)} = \frac{(-2)^2-3(-2)-5}{(-2-1)^3} = \frac{5}{-27} = -5/27$$

To obtain the values of B and C we must **clear fractions in (I)** by multiplying both sides of the equation by $(x-1)^3(x+2)$ to obtain

$$x^2-3x-5 = A(x+2) + B(x+2)(x-1) + C(x+2)(x-1)^2 + D(x-1)^3 \qquad \text{(II)}$$
$$x^2-3x-5 = A(x+2) + B(x^2+x-2) + C(x^3-3x+2) + D(x^3-3x^2+3x-1)$$

Collecting like terms on the right side we obtain:

$$0x^3+x^2-3x-5 = (C+D)x^3 + (B-3D)x^2 + (A+B-3C+3D)x + (2A-2B+2C-D)$$

Since **only two unknows are left** to be determined, we need only to form **two equations** to solve for them. The easiest equations are obtained by equating the coefficients of the cubes and squares of x.

$$C+D=0 \quad \text{(from the cubes)}$$
$$B-3D=1 \quad \text{(from the squares)}$$

Since we already know that **D=-5/27** then we easily obtain from these equations the values **C=5/27** and **B=3(-5/27)+1=12/27.**

Usually when only one or two unknowns are left to be determined, equating the coefficients of the **highest power** of x is the easiest to use (assuming the equation involves at least one of the unknowns yet to be determined). The next easiest is usually equating the **constants**. These can normally be obtained mentally from (II).

Heaviside's cover-up technique is simply a **nice bookkeeping system** for substituting the values 1 and -2 into the cleared equation (II) and solving for **A** and **D** respectively.

The **second** speed-up method works only for fractions whose denominator is a **power** of a **SINGLE** term. For example, this method

will work on $\dfrac{x^2-6x-5}{(x-2)^3}$ but not on $\dfrac{x^2-6x-5}{x(x-2)^3}$;

will work on $\dfrac{2x^4+2x-7}{(x^2+4)^3}$ but not on $\dfrac{2x^4+2x-7}{(x+3)(x^2+4)^3}$.

The method consists of **dividing repetitively** by the factor that is raised to a power (x-2 in the first example; x^2+4 in the second). **Each quotient** is in turn divided by the factor until the quotient is of **degree less than** the factor. The **remainders** of the divsions in the **order** they are obtained are the **numerators** for the decending powers of the factor. The **last quotient** is used **after** the last remainder. It's easier to demonstrate than to explain, so here goes:

$$\frac{x^2-6x-5}{(x-2)^3} = \frac{A}{(x-2)^3} + \frac{B}{(x-2)^2} + \frac{C}{x-2} = \frac{-13}{(x-2)^3} + \frac{-2}{(x-2)^2} + \frac{1}{x-2}$$

$$\begin{array}{r|l} & \underline{x\ \ -4} \\ x-2 & x^2-6x-5 \\ & \underline{x^2-2x} \\ & \quad -4x-5 \\ & \quad \underline{-4x+8} \\ & \qquad -13 \quad \textbf{(This is A.)} \end{array} \qquad \begin{array}{r|l} & \underline{1} \quad \textbf{(This is C.)} \\ x-2 & x-4 \\ & \underline{x-2} \\ & \ -2 \quad \textbf{(This is B.)} \end{array}$$

And the second example follows:

$$\frac{2x^4+2x-7}{(x^2+4)^3} = \frac{Ax+B}{(x^2+4)^3} + \frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{x^2+4} = \frac{2x+25}{(x^2+4)^3} + \frac{-16}{(x^2+4)^2} + \frac{2}{x^2+4}$$

```
          2x^2+ 0x^3- 8                            2            (This is Ex+F.)
        ___________________________          ____________
x^2+4  | 2x^4+ 0x^3+ 0x^2+ 2x - 7      x^2+4 | 2x^2+ 0x - 8
         2x^4      + 8x^2                     2x^2+ 0x + 8
        ________________                      ____________
                -8x^2+ 2x - 7                      0x -16 (This is Cx+D.)
                -8x^2     -32
                ____________
                      2x+ 25  (This is Ax+B.)
```

These divisons can be more **easily and speedily** performed using a generalization of synthetic division (**GenSynD**).

Sometimes the numbers obtained from the successive remainders and the last quotient are **not enough** to produce **all** the numerators needed. In this case the numbers **not produced** by the successive divsions are **all set equal to zero**. A slight revision of the first example illustrates this.

$$\frac{x^2-6x-5}{(x-2)^5} = \frac{A}{(x-2)^5} + \frac{B}{(x-2)^4} + \frac{C}{(x-2)^3} + \frac{D}{(x-2)^2} + \frac{E}{x-2}$$

```
          x -4                         1  (This is C.)
        _________                    ______
x-2    | x^2-6x-5              x-2  | x-4
         x^2-2x                       x-2
         ______                       ____
            -4x-5                      -2  (This is B.)
            -4x+8
            _____
              -13  (This is A.)
```

As before the values of **A, B and C** are obtained from these **two** divisions. But since the degree of the last quotient is less than that of the divisor x-2, we <u>cannot divide any further</u>. The values of **D and E** are therefore **zero.** Thus we obtain the decompositon

$$\frac{x^2-6x-5}{(x-2)^5} = \frac{-13}{(x-2)^5} + \frac{-2}{(x-2)^4} + \frac{1}{(x-2)^3} + \frac{0}{(x-2)^2} + \frac{0}{x-2}$$

Most methods of partial fraction decomposition require that the <u>degree of the numerator be less than the degree of the denominator.</u> This "single factor to a power" situation **works even when** the numeratior is of **equal or greater** degree than the denominator. As an example consider

$$\frac{2x^3-3x^2+x-2}{(x+2)^2} = \frac{A}{(x+2)^2} + \frac{B}{x+2} + P(x)$$

where $P(x)$ is the quotient of the indicated division and the fractions involving **A** and **B** are the decomposition of the **remainder** of the division by $(x+2)^2=x^2+4x+4$.

The repeated division is done as many times as the indicated power of x+2 in the denominator (twice in this case). The two remainders are the values of A and B. The **last quotient** is the polynomial **P(x)**.

```
         2x^2-7x+15                   2x-11        (This is P(x).)
      ---------------              ---------
x+2 | 2x^3-3x^2 + x -2       x+2 | 2x^2-7x+15
      2x^3+4x^2                    2x^2+4x
      ---------                    -------
         -7x^2+  x                   -11x+15
         -7x^2-14x                   -11x-22
         ---------                   -------
              15x- 2                      37   (This is B.)
              15x+30
              ------
                 -32  (This is A.)
```

Therefore the decompositon is:

$$\frac{2x^3-3x^2+x-2}{(x+2)^2} = \frac{-32}{(x+2)^2} + \frac{37}{x+2} + 2x - 11$$

The **third** technique works when we are decomposing into **only two** fractions with different factors in their denominators, **especially** when one of the denominators is a **linear factor.** For example,

$$\frac{1}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}$$

The value of **A** is easily obtained using Heaviside's cover-up technique (We use ::: for the "covering up".)

$$A = \frac{1}{:::(0^2+2*0+2)} = \frac{1}{2}$$

We shall use the cross method $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ of adding fractions to

determine the values of **B** and **C**. The product $A(x^2+2x+2)$ yields an x^2 term of $(1/2)x^2$. Since the sum of the fractions has $0x^2$ in its numerator, the other product $x(Bx+C)$ must produce $(-1/2)x^2$ to cancel $(1/2)x^2$ obtained in the other cross product. Hence, **B** must be **-1/2**. Reasoning in a similar fashion we note that the product $A(x^2+2x+2)$ yields an **x** term of $Ax=(1/2)(2x)=x$. Since the **numerator** of the sum of the fractions has **0x** in it, then this **x** must be **cancelled out** by the **x** term in the product $x(Bx+C)$; that is, **Cx** must produce **-x.**

Hence **C** must be **-1**. The complete decomposition is then

$$\frac{1}{x(x^2+2x+2)} = \frac{1/2}{x} + \frac{(-1/2)x - 1}{x^2+2x+2}$$

As another example consider

$$\frac{2x-7}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}\ .$$

The value for A is obtained by Heaviside's method.

$$A = \frac{2(2)-7}{:::(2+1)} = \frac{-3}{3} = -1.$$

Since the **x** term of the sum is **2x** and the product **A(x+1)** produces **-x**, then the product **B(x-2)** must produce **3x** so that the sum **-x+3x** is **2x**. Hence, **B** must be **3**. (To obtain **B** we could also have used the fact that the sum has a constant term of **-7**; that is **A(x+1)** produces a constant term of **(-1)(1)=-1**, so **B(x-2)** must produce a constant term of **-6**. Hence **B** must be **3**.) Of course, we could have used Heaviside's method to obtain **B**, but this method seems to be just a bit faster.

A one last example consider

$$\frac{2x^3-7x^2-4x+1}{(x+1)(x-1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}\ .$$

By Heaviside's method we obtain **A and B**:

$$A = \frac{2(-1)^3-7(-1)^2-4(-1)+1}{:::(-1-1)((-1)^2+1)} = 1 \quad \text{and} \quad B = \frac{2(1)^3-7(1)^2-4(1)+1}{(1+1):::(1^2+1)} = -2.$$

Then we **add the first two fractions** back together to obtain

$$\frac{2x^3-7x^2-4x+1}{(x+1)(x-1)(x^2+1)} = \frac{-x-3}{x^2-1} + \frac{Cx+D}{x^2+1}$$

Now with just two fractions we can use this third method. The sum has a constant term of **1** and the product **(-x-3)(x²+1)** produces **-3**. So the product **(x²-1)(Cx+D)** must produce **4**. Hence **D=-4**. The sum has an **x** term of **-4x** and the product **(-x-3)(x²+1)** produces **-x**. So the product **(x²-1)(Cx+D)** must produce **-3x**. Hence **C** must be **3**. The complete decomposition is therefore

$$\frac{2x^3-7x^2-4x+1}{(x+1)(x-1)(x^2+1)} = \frac{1}{x+1} + \frac{-2}{x-1} + \frac{3x-4}{x^2+1}$$

Note: We used the constant and x terms to find **C** and **D**. We could just as well have used the x squared and x cubed terms.

Here is **one more example** of the **repeated division technique** that (together with the shorter division method) illustrates its power. **Using the standard method** of clearing fractions and equating coefficients would require setting up and solving **six equations** in **six unknowns**. This would require about a legal size sheet of paper to work it out. On the other hand using a **generalization of synthetic division** (GenSynD) ...

$$\frac{x^5-2x^4+3x^3-x^2+2x-3}{(x^2-x+2)^3} = \frac{Ax+B}{(x^2-x+2)^3} + \frac{Cx+D}{(x^2-x+2)^2} + \frac{Ex+F}{x^2-x+2}$$

```
                1 -2                                1 -2

             1 -1  0  1 |                        1  0 |
           ------------------                ----------------
1 -1  2  |  1 -2  3 -1 |  2 -3       -1  2  | 1 -1 |  0  1
               1 -3  2 |  1 -2                   1 | -2  0
            ------------------                ----------------
             1 -1  0  1 |  3 -5                1  0 | -2  1
```

	Short form	= Expanded form	= the required
First remainder:	**3 -5**	**3x-5**	**Ax+B**
Second remainder:	**-2 1**	**-2x+1**	**Cx+D**
Last (2nd) quotient:	**1 0**	**1x+0=x**	**Ex+F**

Sometimes we can use the repeated division technique on part of a problem as in the following example in which we "peel off" the extra factor x so that we can get at the rest of the problem.

$$\frac{x^2-6x-16}{x(x-2)^3} = \frac{A}{x} + \frac{P(x)}{(x-2)^3} \quad \text{where} \quad \frac{P(x)}{(x-2)^3} = \frac{B}{(x-2)^3} + \frac{C}{(x-2)^2} + \frac{D}{x-2}$$

We can solve for A very easily by **Heaviside's**: $A = -16/(-2)^3 = 2.$

Then clear fractions to obtain

$$x^2-6x-16 = A(x-2)^3 + xP(x)$$

$$x^2-6x-16 = 2(x^3-6x^2+12x-8) + xP(x).$$

Then we solve for P(x):

$$x^2-6x-16 = 2x^3-12x^2+24x-16 + xP(x)$$

$$-2x^3+13x^2-30x = xP(x)$$

$$P(x) = -2x^2+13x-30$$

Now we can find **B, C and D** using the repeated division method:

```
              2                                2
           -2   9 |                          -2 |     (This is D.)
          ---------------                  ---------
  1 -2 |   -2  13 | -30             1 -2 |  -2 |  9
                -4 |  18                        | -4
          ---------------                  ---------
           -2   9 | -12  (This is B.)        -2 |  5   (This is C.)
```

So we now have

$$\frac{x^2-6x-16}{x(x-2)^3} = \frac{2}{x} + \frac{-12}{(x-2)^3} + \frac{5}{(x-2)^2} + \frac{-2}{x-2}$$

This particular problem could have been done perhaps a little easier by solving for both **A** and **B** by Heaviside's technique and then clearing fractions, etc. to solve for **C** and **D**. However, if the cubed factor had been something like $(x^2+4)^3$ or $(x^2+3x+5)^3$, then there would be a definite advantage to this "**peeloff**" technique.

Most **college algebra books** can provide lots of problems to practice on and usually about half of them have answers.