SUBTRACTION BY SUBTRACTING FIRST THEN BORROWING

The method indicated in the title is much **faster** and **easier** than the traditional "Borrow and then subtract." We will list some of the advantages later, but for now let's see how it is done.

There are **two** lines to be written, but each line is **quite easy** to do and requires **little** "mental conversation" to do so. We will start with some simple examples and then show how to extend the idea to larger problems. Consider the first line that we need to write in the subtraction

$$\begin{array}{r} 637 \\ -\underline{257}. \end{array}$$

Step 1) If the digits are the **same** we just write down 0(zero).
2) If the digits are **not** the same then we **always** record the result of the **larger** digit less the **smaller** digit.
3) ***If*** the larger digit is on the **bottom**, put a "**bar**" over the result.

In obtaining the first line of the subtraction each column is **independent** of the other columns so we have three **separate** calculations to do.

In the example above the units digits are both 7 so we write 0(Step 1). Then the tens digits are 3 and 5 so we record the difference 2 and **put a bar over the 2** since the **larger** digit is on the **bottom** (steps 2 and 3). In the hundreds column we record the result of 6-2 and **put no bar** over the 4 since the **larger** digit was on top (step 2).

Thus we obtain $\begin{array}{r} 637 \\ -\underline{257} \\ 4\bar{2}0 \end{array}$ which gives us a correct answer but it is just not the form we are most familiar with. It represents 400 less 20 which is 380.

Going from the $4\bar{2}0$ to the 380 is called "debarring" and will be examined in more detail below. But first we will do a few more examples of the first line.

36485	74304	769345	8000321	85975
-18951	-31561	-274317	-2473859	-23835
$2\bar{2}\bar{5}34$	$43\bar{2}\bar{6}3$	$5\bar{1}503\bar{2}$	$6\bar{4}\bar{7}\bar{3}\bar{5}\bar{3}\bar{8}$	62140

Each of these are worked by following the three simple rules above. Also note that each column is done **independently** so that we may proceed from left to right or right to left (or just skip around if we want to).

The barred digits are read zerobar, onebar, twobar, etc. So the number $34\bar{2}7\bar{3}\bar{4}$ is read "three four twobar seven threebar fourbar."

To get the second (and more familiar) line of the answer we go from **left to right** since there is some interaction between the columns due to borrowing. Notice that the last example **doesn't need debarring** since it <u>has no bars</u> at all.

Any two **digits** that add to 9 are called <u>nines complements</u>: 0&9, 1&8, 2&7, 3&6 and 4&5. The <u>tens complements</u> are 1&9, 2&8, 3&7, 4&6 and 5&5. These are used in debarring.

A basic <u>**debar sequence**</u> consists of a **positive** digit followed <u>immediately</u> by **one or more** <u>*consecutive*</u> barred digits.

Examples: $6\bar{3}$, $4\bar{8}\bar{2}$, $9\bar{4}\bar{7}\bar{3}$, $8\bar{8}\bar{1}\bar{6}\bar{5}$, $5\bar{7}\bar{9}\bar{5}\bar{6}\bar{1}$

Debarring is done by the following steps:

Step 1) Decrease the positive digit by one.
Step 2) Write the <u>tens complement</u> of the **rightmost** bar digit.
Step 3) Write the <u>nines complement</u> of the **rest** of the bar digits (if there are others).

We **decrease** the positive digit by **one** because we are **borrowing** from it. For the examples above we obtain:

57 318 8527 71835 420439.

<u>**Zeros**</u> deserve special attention since they <u>may or may not</u> need to be considered barred. **Initially** consider <u>**all**</u> the zeros in a number to be **not** barred. Then ***IF*** the digit to the <u>**immediate right**</u> of a zero is a barred digit, put a bar on the zero.

Examples: $40\bar{3}$, $2\bar{5}0\bar{8}$, $70\bar{3}0\bar{8}$, $6\bar{7}\bar{8}0\bar{2}0\bar{6}$ become

$4\bar{0}\bar{3}$, $2\bar{5}\bar{0}\bar{8}$, $7\bar{0}\bar{3}\bar{0}\bar{8}$, $6\bar{7}\bar{8}\bar{0}\bar{2}\bar{0}\bar{6}$ which debarred are

397, 1492, 69692, 5219794.

If two or more zeros occur in a row (<u>consecutively</u>) and the digit to the **immediate right** of the <u>rightmost</u> zero is barred, then <u>**ALL**</u> the consecutive zeros require a bar.

Examples: $300\bar{4}$, $7\bar{5}000\bar{3}\bar{4}$, $80\bar{4}00\bar{5}\bar{6}000\bar{9}$ become

$3\bar{0}\bar{0}\bar{4}$, $7\bar{5}\bar{0}\bar{0}\bar{0}\bar{3}\bar{4}$, $8\bar{0}\bar{4}\bar{0}\bar{0}\bar{5}\bar{6}\bar{0}\bar{0}\bar{0}\bar{9}$ which debarred are

2996, 6499966, 79599439991.

Other zeros remain zeros.

Examples: $503\bar{4}0$, $3\bar{4}07$, $6\bar{8}\bar{4}00$, $6043\bar{0}\bar{0}\bar{3}$, $7\bar{3}40$, $7\bar{0}\bar{4}02$
equal 50260, 2607, 51600, 6042997, 6740, 69602.

A number to be debarred <u>may have **more than one**</u> debar sequence. Then each debar sequence is debarred individually.

Example: $2\bar{3}4\bar{7}$ has two debar sequences $2\bar{3}$ and $4\bar{7}$. We get 1733.

Example: $74\bar{2}\bar{6}503\bar{7}\bar{0}\bar{6}8$ has debar sequences $4\bar{2}\bar{6}$ and $3\bar{7}\bar{0}\bar{6}$ which give 73745022948

Note that the first zero (between 5 and 3) does not require a bar but the second zero (to the immediate left of sixbar) does require it.

With a bit of practice debarring becomes pretty "automatic." Then we can do subtraction by "subtracting first and borrowing later" instead of "borrowing first and subtracting later." This makes subtraction a "piece of cake" compared to the laborious "borrowing" process normally taught in school.

<u>Advantages of this "Op and Add" subtraction</u>

1) Each of the two lines are easy and can be done rapidly.
2) The entire process requires no addition/subtraction combinations in excess of 10 (no 11's through 18's).
3) This is how subtraction is done in algebra (the first line) so the transition to algebra is a little smoother. (More on this in a future article)
4) It introduces the bar notation which becomes increasingly more useful in terms of making expressions more readable and user friendly in algebra.
5) Accuracy tends to be better using this technique.

HISTORICAL NOTE:

A person named Colson introduced this kind of subtraction in 1729 in a paper entitled **"Positivo-Negativo Arithmetic."** It was pretty much lost in the archives until Cedric Smith found the paper and included a special chapter on this in a book he was writing entitled **"Biomathematics"** in 1927. This didn't make much of a splash either. In 1967 a fourth grader (first name Kye) "reinvented the wheel" so to speak in a school classroom. Since then it has received a little attention, but not nearly enough.

It seems that none of these folks noticed that bar digits can be useful in multiplication as well. For example to multiply 199*198 we can write $20\bar{1}$*$20\bar{2}$ and multiply:

$$\begin{array}{rr}
 & 20\bar{1} \\
 * & \underline{20\bar{2}} \\
 & \bar{4}02 \\
 & 000 \\
 & \underline{40\bar{2}} \\
 & 4\bar{0}\bar{6}02 \\
 = & 39402
\end{array}$$

So we get to use **much smaller digits** in the multiplication by rewriting (**enbarring**) the 199 and 198. You might have a bit of fun :) playing around with this concept. You can actually get by with no product larger than 5*5 if you use enbarring adroitly. The rules for enbarring are quite analogous debarring rules.

In the early 1970's the author of this little paper wrote a book entitled **"Bar Arithmetic"** which showed how to use this bar notation to considerably speed up addition, subtraction, multiplication, division and square root calculations by hand. But alas! there was no marketing interest for such a book since hand calculators were coming on the market at that time. Talk about **bad timing**! :(

I've see about 20 different ways to do subtraction (a very versatile operation) and of all those ways it appears that the classical **"borrowing"** technique **is** easily the **most difficult** and **slowest** of them all. You might be amazed if you write down **all** that you **say** in your mind when you do about six columns of borrowing in a subtraction problem. Some of my students did so and wrote nearly a whole page. No wonder it is so slow!

Back in the 1970's I showed a fourth grade teacher how to do this subtraction and she in turn taught her fourth grade students how to do it. **They loved it!**

In case you are wondering how I get the bars over the digits, I am using Word Perfect with Courier New (True Type) with .5 spacing. Word Perfect lets you space in **tenths of a space** using the **line spacing** command. You can get to it by the sequence: **Format --> Line --> Spacing**. A menu will appear that will allow you to adjust the spacing in tenths of a line of single spacing. If two lines **overlap** because of small spacing, then **both** lines will show and print. This can be very handy! I have not seen any other commercial word processor that can do this. WP also has an **overstrike** feature that allows you to print two or more characters in the **same** space. For example from 0 and / we can get Ø. We get ∀ from V and -. From C, / and underline we get ⊄ for "not a subset". ≤ is from < and underline. There are lots of possibilities for creating symbols without having to go to a special font. All of this is in **Courier New TT** font.

*In **Lucida Sans Typewriter** the same group of symbols is Ø, ∀, ⊄, and ≤.*
*In **Courant TT** the same group of symbols is Ø, ∀, ⊄, and ≤.*
In Consoles TT the same group of symbols is Ø, ∀, ⊄, and ≤.
In monospaced TT the same group of symbols is Ø, ∀, ⊄, and ≤.
In **Letter Gothic** the same group of symbols is Ø, ∀, ⊄, and ≤.
In Lucida Console TT the same group of symbols is Ø, ∀, ⊄, and ≤.

These are most of the monospaced fonts; that is, they are **not proportional** so that each letter takes up the **same** space. This is critical for typing the math since super and sub scripts must **line up** with the characters they are supposed to be on.

NAME:_________________ DATE:_________ BarSubt2.wpd

Subtract using bars (Opp and Add):

7727 4325	81793 71939	550 499	978699 302121	81544485 81174649
4779 4706	83388 73969	936 869	399177 124699	52052669 28421465
5414 4772	77113 44129	528 222	649091 642437	77942577 16992343
7496 3764	96373 56331	599 580	924611 892767	53393139 26461665
5857 3514	51797 14142	511 479	839726 766252	88753646 53867776
8664 3439	62233 22590	427 182	845357 292622	93247609 68664630
8494 7309	83379 55211	945 862	854124 685951	79709462 45587669

NAME:_________________ DATE:__________ BarSubt2.wpd

Subtract using bars (Opp and Add): **Answer Key**

7727	81793	550	978699	81544485
4325	71939	499	302121	81174649
3402	$1\bar{0}\bar{2}6\bar{6}$	$1\bar{4}\bar{9}$	676578	$4\bar{3}\bar{0}\bar{2}4\bar{4}$
	9854	51		369836
4779	83388	936	399177	52052669
4706	73969	869	124699	28421465
73	$1\bar{0}\bar{6}2\bar{1}$	$1\bar{3}\bar{3}$	$275\bar{5}\bar{2}\bar{2}$	$3\bar{6}\bar{4}31204$
	9419	67	274478	23631204
5414	77113	528	649091	77942577
4772	44129	222	642437	16992343
$1\bar{3}\bar{6}2$	$33\bar{0}\bar{1}\bar{6}$	306	$7\bar{4}6\bar{6}$	$61\bar{0}\bar{5}0234$
642	32984		6654	60950234
7496	96373	599	924611	53393139
3764	56331	580	892767	26461665
$4\bar{3}32$	40042	19	$1\bar{7}2\bar{1}\bar{5}\bar{6}$	$3\bar{3}\bar{1}32\bar{5}\bar{3}4$
3732			31844	26931474
5857	51797	511	839726	88753646
3514	14142	479	766252	53867776
2343	$4\bar{3}655$	$1\bar{6}\bar{8}$	$1\bar{3}35\bar{3}4$	$35\bar{1}\bar{1}\bar{4}1\bar{3}0$
	37655	32	73474	34885870
8664	62233	427	845357	93247609
3439	22590	182	292622	68664630
$523\bar{5}$	$4\bar{0}\bar{3}\bar{6}3$	$3\bar{6}5$	$6\bar{5}3\bar{3}35$	$3\bar{5}\bar{4}\bar{2}3\bar{0}\bar{3}9$
5225	39643	245	552735	24582979
8494	83379	945	854124	79709462
7309	55211	862	685951	45587669
$119\bar{5}$	$3\bar{2}168$	$1\bar{2}3$	$2\bar{3}\bar{1}\bar{8}\bar{3}3$	$342\bar{8}2\bar{2}\bar{0}\bar{7}$
1185	28168	83	168173	34121793